A NEW STABLE TAYLOR-HOOD FINITE ELEMENT IN LINEAR ELASTICITY

Dubravka Mijuca

Abstract — In the present paper a new stable [1] three-dimensional finite element based on the primal-mixed formulation in elasticity, is presented. According to the known literature, it is the first reliable *mixed* finite element in 3d elasticity, where *a priori* C^0 continuous discretization for stresses and displacements, as fundamental variables is used. It has properties of the up most importance for the successful applications on the real structures – it satisfies consistency, ellipticity and inf-sup conditions, which ensure good mathematical convergence characteristics. In addition, it is provided that there are no difficulties in the numerical solution of the resulting linear system of equations. It should be noted that present formulation contains no tricks, no tune-ups, or hidden features. The main objective for the present investigation was identification of an universal finite element in elasticity that can replace library of finite elements for specific purposes presently used in finite element software codes.

Key words — Linear Elasticity, Finite Element Method, Numerical stability

1. Introduction

The mixed finite element schemes [2] are now abundantly used for the analysis of fluid flows, almost incompressible and incompressible materials, plates and shells. In the number of papers [3,4,5] it has been shown that special class of mixed finite element methods, so called primal-mixed method, should be employed in the plane elasticity analysis, rather than displacement method. Recently, an extension to three-dimensional settings was made [6,7] relaying on a full 3d theory. It has been shown that the lowest-order finite element HC8/9 of that scheme is solvable and robust, i.e. it is not sensitive to changing of Poisson's ratio. Further it has been shown that it satisfies ellipticity condition – the first stability condition, since the bilinear form connected to the stress space is coercive. Moreover, it has been shown that it can be used as a universal finite element applicable in the analysis of beams, rods, plate and shells, full bodies of arbitrary geometry in analysis of compressible and almost incompressible materials that are in the class of regular model problems (e.g. subjected to the smooth boundary conditions). However, it has been numerically proven that it is not stable, i.e. it does not pass numerical inf-sup test. That results was expected, since its two-dimensional counterpart does not pass that test also, although it was numerically proven that it is very efficient and it has been recommended [6] for the analysis of regular model problems.

Here will be shown that is possible to construct reliable [1] finite element method in linear elasticity that automatically satisfies the first and second stability condition and at the same time has C^0 continuous stresses. More clearly, this paper is an answer to the paper of Brezzi *et* al. [8] where several observations concerning mixed finite element schemes are presented. Firstly, "...for linear elasticity problems such a construction (satisfying the first and second stability condition) is yet unachieved and looks rather difficult". Secondly, there is a remark about so-called Taylor-Hood mixed finite element in elasticity where both variables of interest (stresses and displacements) are continuous "…It is not known if this element is stable". And finally, "…The use of C^0 discretization for the stress filed should be avoided. The main reason for this is the difficulty in the numerical solution of the linear system of equations."

Consequently, a new finer continuous finite element approximation spaces are used for the discretization of the present underlying formulation. On that way, a new mixed three-dimensional Taylor-Hood finite element in elasticity, named HC8/27, is obtained. In order to check weather it is reliable or not, it is investigated for its mathematical convergence characteristics [1]. As a result, it has been shown that it satisfies consistency, ellipticity and inf-sup conditions. In addition it is shown that it is solvable and robust. In the present paper, the results of the inf-sup stability test will be presented, only. It should be noted also, that resulting system of linear algebraic equations, where system matrix is indefinite, symmetric, and sparse, is without difficulties solved by direct Gaussian elimination procedure.

2. Present Formulation

Present finite element method was firstly investigated in plane elasticity settings in early eighties. Through the time, changes in chosen finite element shapes and corresponding local basis functions shapes, type of solution technique and type of boundary conditions, were made. Eventually, a reliable and time efficient scheme in plane elasticity, is obtained [5,6], in spite of additional three equations per global node connected to the stress components, for difference to displacement method.

In the paper [5] it is explained why finite element displacement method in elasticity, in its original form is not applicable to problems of thin plates and shells and high Poisson's ratio. Nevertheless, as a cure for these situations, artificial constants are usually introduced in the displacement local functions expressions or reduced integration is used, which directly makes this formulation unreliable [1]. In addition, in the displacement method, stresses are calculated in an additional procedure on local element level, resulting in unrealistic discontinuity of stresses along the finite element boundaries, for the model problems without singularities.

After proving present formulation in two-dimensional settings, its extension to three-dimensional settings was made and results were published in [7]. In that paper the finite element HC8/9 is investigated in detail, where it was shown that it is solvable and robust, but not stable. These results were in exact accordance with the results obtained for its two-dimensional counterparts QC4/5. It should be noted, that in the present literature it was the first time that numerical stability of mixed finite element approach in the three-dimensional setting was determined.

In the present paper, the new finite element HC8/27 is introduced where stress spaces are even more refined. It will be shown that this element satisfies the first and second stability condition, known as an ellipticity on the kernel condition and inf-sup condition, respectively.

Though the underlying theory can be found in number of papers about the primal-mixed formulation [9], for the sake of clarity its basic expressions are given below.

The present procedure is based on the Hellinger–Reissner variational principle [10] in linear elasticity, for which the weak form of primal-mixed variational formulation with displacement and stresses used from continuous space functions, is given by:

$$\text{Find } \{\boldsymbol{u},\boldsymbol{T}\} \in H^1(\Omega)^n \times H^1(\Omega)^{n\times n}_{\text{sym}} \text{ such that } \boldsymbol{u}\big|_{\partial\Omega_u} = \boldsymbol{w} \text{ and:}$$

$$\int_\Omega (\boldsymbol{AT} : \boldsymbol{S} - \boldsymbol{S} : \nabla\boldsymbol{u} - \nabla\boldsymbol{v} : \boldsymbol{T})d\Omega = -\int_\Omega \boldsymbol{v}\cdot\boldsymbol{f}d\Omega - \int_{\partial\Omega_t} \boldsymbol{v}\cdot\boldsymbol{p}d\partial\Omega\,, \tag{1}$$

$$\text{for all } \{\boldsymbol{v},\boldsymbol{S}\} \in H^1(\Omega)^n \times H^1(\Omega)^{n\times n}_{\text{sym}} \text{ such that } \boldsymbol{v}\big|_{\partial\Omega_t} = \mathbf{0}.$$

In this expression $\boldsymbol{u}$ is displacement field, $\boldsymbol{T}$ is the stress field, $\boldsymbol{f}$ is the vector of body forces and $\boldsymbol{p}$ is the vector of surface forces. Further, $\boldsymbol{A} = \boldsymbol{K}^{-1}$ is the elastic compliance tensor, while $\boldsymbol{v}$ and $\boldsymbol{S}$ are the displacement and stress weight functions, respectively. Space $\boldsymbol{T} \in H^1(\Omega)^{n\times n}_{sym}$ is the space of all symmetric tensorfields that have square integrable gradient, while space $H^1(\Omega)^n$ is the space of all vectorfields that are square integrable and have square integrable gradients, where n is the number of spatial dimensions of the problem under consideration

It has been shown in [3] that present scheme can be decomposed for unknown (variable) and known (prescribed) values of the stresses and displacements denoted by the indices v and p respectively:

$$\begin{bmatrix} \mathbf{A}_{vv} & -\mathbf{D}_{vv} \\ -\mathbf{\Delta}_{vv}^{T} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{t}_v \\ \mathbf{u}_v \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{vp} & \mathbf{D}_{vp} \\ \mathbf{D}_{pv}^{T} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{t}_p \\ \mathbf{u}_p \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_p + \mathbf{p}_p \end{bmatrix}. \tag{2}$$

In these expressions, the nodal stresses $t^{L\,st}$ and displacements u_{Kq} components are consecutively ordered in the column matrices **t** and **u** respectively. The members of the matrices **A** and **D** and of the vectors (column matrices) **f** and **p** (discretized body and surface forces) are respectively:

$$A_{\Lambda uv\Gamma st} = \sum_e \int_{\Omega_i} \Omega^N_\Lambda S_N \, g^a_{(\Lambda)u} g^b_{(\Lambda)v} A_{abcd}\, g^c_{(\Gamma)s} g^d_{(\Gamma)t} T_L \Omega^L_\Gamma d\Omega; \tag{3}$$

$$D^{\Gamma q}_{\Lambda uv} = \sum_e \int_{\Omega_i} \Omega^N_\Lambda S_N U^K_a \Omega^\Gamma_K g^a_{(\Lambda)u}\, g^{(\Gamma)q}_{(\Lambda)v} d\Omega, \tag{4}$$

$$f^{\Lambda q} = \sum_e \int_{\Omega_i} g^{(\Lambda)q}_a \Omega^\Lambda_M V^M f^a d\Omega, \qquad p^{\Lambda q} = \sum_e \int_{\partial\Omega_{it}} g^{(\Lambda)q}_a \Omega^\Lambda_M V^M p^a d\partial\Omega \tag{5}$$

From the Eq. (2) it can be seen that present system of equation is symmetric and indefinite. Further, system matrix will have positive and negative eigenvalues due to the fact that the saddle point problem is discretized.

3. The HC finite element family

The HC finite element family is shown in Fig.1. In the present notation, letter H stands for element hexahedral geometry, while letter C indicates continuous interpolation of displacement and stress fields. These letters are followed by number of displacement and stress local nodes per element, respectively. Circles depict displacement nodes, while stress nodes are represented by tetrahedrons. The basic properties of the finite elements HC8/8 and HC8/9, may be found in [7]. The present finite element HC8/27 has 3 and 6 degrees of freedom, per displacement and stress node, respectively. Both fields are approximated by the tri-linear interpolation functions. In addition, stress field is enriched by quadratic hierarchic shape functions connected to the additional 21 hierarchic nodes. The numerical integration is performed by 3×3×3 numerical Gaussian integration. It should be noted that, besides additional degrees of freedom, in the two-dimensional case it was shown that this formualtion has better time/accuracy ratio than displacement method [5,6].

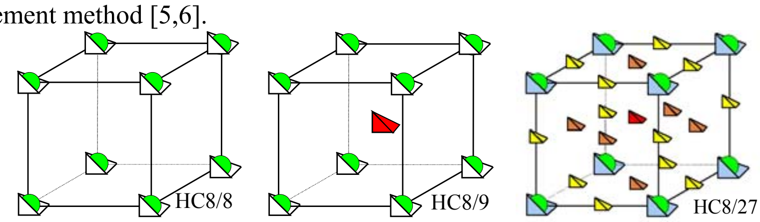

Figure 1. The HC finite element family.

4. The Stability Test

The finite element is stable if it satisfy two necessary conditions i.e., the first condition represented in the *ellipticity on the kernel* condition and second condition represented in the *inf-sup* condition [1]. In the present case, the test and trial stress local functions are from space $\left(H^1\right)^{n\times n}$ that ensures that corresponding bilinear form, connected to the test and trial stress approximate spaces, is coercive. From that reason, the first stability condition [9] is automatically satisfied, like in for example, Stokes problems.

The second condition for stability is satisfied if for the meshes of increasing density, value γ_h, following from LBB (Ladyzhenskaya, Babuška, Brezzi) condition and defined in e.g. [11], p.76, Eq.(3.22), remains bounded above zero,

$$\gamma \le \gamma_h = \inf_{\mathbf{u}\in H^1} \sup_{\mathbf{T}\in L^2} \frac{b\left(\mathbf{T}_h, \mathbf{u}_h\right)}{\left\|\mathbf{T}_h\right\|\left\|\mathbf{u}_h\right\|}, \tag{6}$$

$$b\left(\boldsymbol{T}_h, \boldsymbol{u}_h\right) = \sum_e \int_{\Omega_i} \boldsymbol{T}_h : \nabla \boldsymbol{u}_h d\Omega_e , \quad \left\|\boldsymbol{T}_h\right\|^2 = \sum_e \int_{\Omega_i} \boldsymbol{T}_h : \boldsymbol{T}_h d\Omega, \quad \left\|\boldsymbol{u}_h\right\|^2 = \sum_e \int_{\Omega_i} \nabla \boldsymbol{u}_h{}^{\mathbf{T}} : \nabla \boldsymbol{u}_h d\Omega \tag{7}$$

Because verification of condition like (6) involves an infinite number of meshes, a numerical inf-sup test should be performed for a sequence of several meshes of increasing refinement [1]. Consequently, in the present case, numerical *inf-sup* test in matrix notations is stated as the generalized eigenvalue problem given by:

$$\mathbf{D}^{T}\mathbf{A}^{-1}\mathbf{D}\,\boldsymbol{x} = \lambda\,\mathbf{K}\,\boldsymbol{x}\,, \qquad (8)$$

where **D** and **A** are matrix entries in (2), matrix **K** is the stiffness matrix from the relating displacement finite element method, and $\sqrt{\lambda_{\min}}$ is equal to the inf-sup value γ_h in Eq.(6). This approach is already used in [6] and [7].

Throughout the test the simple unit square block or thin plate, with constrained component displacements along coordinate axes only, is used. The model problem is gradually refined using meshes of $N\times N\times N$ elements, for N=1,2,3. Note that any loading does not enter the test [1]. The obtained inf-sup test results for HC finite element family, presented in Figures 2 and 3, show that finite element HC8/27 is stable only. We must keep in mind that this element, is solvable, i.e. it contains no spurious pressure modes, which will be published elsewhere.

It is interesting to see that if the L^2 norms are used, for which any material characteristics do not enter the test, for determining the eigenvalues in (8) as recommended by Bathe in [1], the similar eigenvalues were obtained, as can be seen from Figure 4. Thus, in the present paper the L^2 norms were obtained using E=1 and ν=0 for calculating the coefficients of matrix **A** and **K,** as it can bee seen in [5].

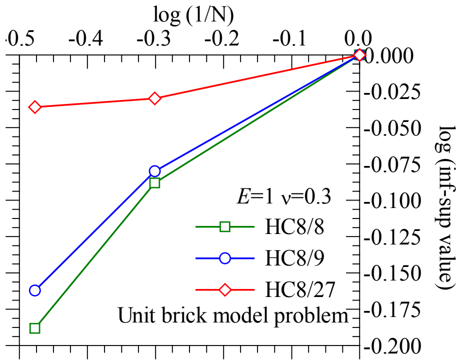

Figure 2. Inf-sup results HC Family.

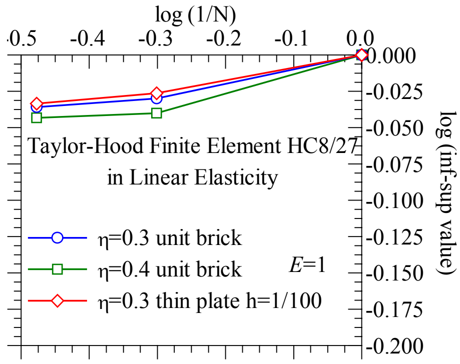

Figure 3. Inf-sup results HC8/27.

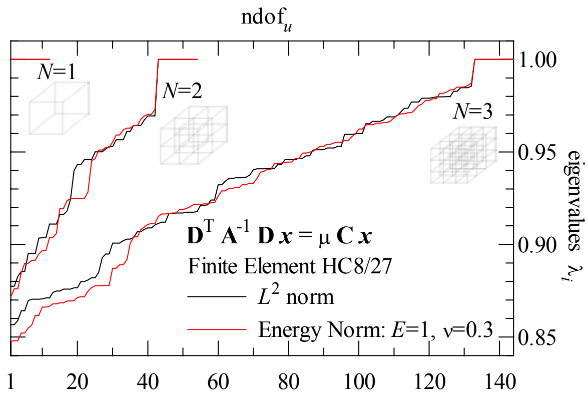

Figure 4. Eigenvalues obtained by inf-sup test (8) for L^2 and Energy norms.

5. Conclusion

In the present paper the stability of the new primal-mixed three-dimensional finite element HC8/27, so-called Taylor-Hood finite element, where displacements and stresses shape functions are *a priori* continuous fundamental variables, was proven. This element is stable as it automatically satisfies the first stability conditions, known as an *ellipticity on the kernel* condition, and passes the second stability test, known as numerical inf-sup test.

Since, the present method is based on the proper variational formulation, in addition it is solvable, and satisfies stability conditions, it can be said that this finite element is reliable and that this finite element scheme may be recommended for general use.

References

1. Bathe K.J., The inf-sup condition and its evaluation for mixed finite element methods. Computers &Structures, 79, 2001, 243-252.
2. Brezzi F.and Fortin M., Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, 1991.
3. Berković M. and Drašković Z., A two-field finite element model related to the Reissner's principle, Theoretical and Applied Mechanics, 20, 1994, 17-36.
4. Berković M., Drašković Z. and Mijuca D., A direct block sparse solution of the mixed finite element equations, Computer Assisted Mechanics and Engineering Sciences, 5, 1998, 21–30.
5. On the main properties of the primal-mixed finite element formulation, with M. Berković, Facta Universitatis Series Mechanics, Automatic Control And Robotics, Vol.2, No 9, 1999, 903–920.
6. Mijuca D. Primal-mixed finite element approach in solid mechanics, Ph.D. Thesis, Faculty of Mathematics, Belgrade (in Serbian), 1999.
7. Mijuca D., A new primal-mixed 3^d finite element, Facta Universitatis Series Mechanics, Automatic Control And Robotics, Vol.3, No 11, 2001, 167-178.
8. Brezzi F., Fortin M. and Marini D., Mixed finite element methods with continuous stresses. Mathematical Models and Methods in applied sciences, Vol.3, No.2, 1993, 275-287.
9. Arnold D.N., Mixed finite element methods for elliptic problems, Computer Methods in Applied Mechanics and Engineering, No. 82, 1990, 281-300.
10. Wempner G., Mechanics of solids with applications to thin bodies, McGraw–Hill, 1973.
11. Brezzi F.and Fortin M., Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, 1991.

Dr Dubravka Mijuca
Faculty of Mathematics, Department of Mechanics, University of Belgrade,
Studentski trg 16, P.0. Box 550, 11000 Belgrade, Yugoslavia
e-mail: dmijuca@matf.bg ac.yu