Displacement based continuous stress recovery procedure,
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Abstract

In this paper the problem of a finite element stress recovery is considered. First, a new global coordinate independent approximation of a continuous stress field is presented. It has been shown that the proposed, FEDSS (Finite Element Displacement type Stress Smoothing) method is computationally more efficient, at least compared with the classical stress averaging procedure. Second, there is a numerical evidence that, at least for four noded isoparametric elements, any stress recovery procedure is less accurate in *strain energy* than direct FEA (Finite Element Analysis). It has also been shown that, for bilinear isoparametric elements, the relative energy error norm with respect to the exact solution, computed by 1×1 Gaussian integration is smallest for the raw FEA, compared with any other stress recovery procedure and any type of the numerical quadrature. Hence, one can recommend (under)integration in the midpoint of an element, i.e. in the derivative (stress) superconvergent point, when error indicators of Z–Z (Zienkiewicz–Zhu) type [1] are calculated.

1 Introduction

Accurate stress prediction is of crucial importance in the analysis and design procedures of real physical bodies. The quality of a solution in the finite element analysis can be improved by the mesh refinement. However, increase of the mesh density drastically raises the solution time. Also, classical displacement type finite element analysis generates continuous displacement field, but discontinuous stress field over the model. Obviously, an acceptable stress recovery procedure should deliver accurate enough continuous stresses over the model in a reasonable time. In addition, derivative recovery techniques are used for error estimates in adaptive finite element procedures.

There are two general classes of the stress smoothing procedures [1, 2, 3]. If carried out over a whole finite element domain the procedure is known as a global smoothing. Local smoothing is performed at each node or small group of nodes.

One of the simplest local smoothing procedures is the averaging of stresses from neighbouring elements at a particular common node. For the purpose of the present paper the method is named FEAavrg.

However, all these approaches including also innovative [4] ones, are based on the conventionally calculated stresses. Hence, all known approaches are scalar, which means that they approximate only one stress component at a time.

It has been shown in [5] that to maintain invariance of the finite element approximations under the coordinate transformations, tensorial character of these approximations should be strictly respected. A significant novelty of the proposed procedure should be pointed out – it is both global and tensorial. These two features, along with high accuracy and computational efficiency, are combined in the present FEDSS method. The second novelty introduced, which is albeit applicable in any Z–Z type error estimator is aforementioned underintegration of error indicator.

2 FEDSS method – basic relationships

Proposed FEDSS method can be used as a stress smoothing procedure as well as an error estimator. At variance with the known approaches, it is based not on the conventionally calculated stresses, but on the displacements. This technique is inspired with some results in [6], related to the two-field finite element method. Nevertheless, it has be shown that similar ideas can be applied in the stress recovery problems.

2.1 Tensorial formulation

In [7] starting point was the stress–displacement relationship:

$$\boldsymbol{A}:\boldsymbol{t} = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right), \qquad \textbf{(1)}$$

where $\boldsymbol{t}$ is the stress tensor, $\boldsymbol{u}$ the displacement vector, and $\boldsymbol{A}$ the elastic compliance tensor.

A weak solution of (1) one can get by the Galerkin procedure using stress variations as the test functions. Finite element approximations are introduced similarly as in [7]. After recasting of all quantities in the coordinate form, one obtains the relationship:

$$A_{\Lambda st\Gamma uv} t^{\Gamma uv} = B^{\Gamma q}_{\Lambda st} u_{\Gamma q} \qquad \textbf{(2)}$$

where, for an element e:

$$\overset{(e)}{A}_{\Lambda st\Gamma uv} = \overset{(e)}{\Omega}{}^{L}_{\Lambda}\, \overset{(e)}{\Omega}{}^{M}_{\Gamma} \frac{\partial z_i}{\partial x^{(\Lambda)s}} \frac{\partial z_k}{\partial x^{(\Lambda)t}} \frac{\partial z_m}{\partial x^{(\Gamma)u}} \frac{\partial z_o}{\partial x^{(\Gamma)v}} \times \int_{\Omega_e} P_L P_M A^{efgh} \frac{\partial z^i}{\partial \xi^e} \frac{\partial z^k}{\partial \xi^f} \frac{\partial z^m}{\partial \xi^g} \frac{\partial z^o}{\partial \xi^h} dV, \qquad \textbf{(3)}$$

$$\overset{(e)}{B}{}^{\Gamma q}_{\Lambda st} = \overset{(e)}{\Omega}{}^{L}_{\Lambda}\, \overset{(e)}{\Omega}{}^{\Gamma}_{K} \int_{\Omega_e} P_L P_a^K \frac{\partial z_i}{\partial x^{(\Lambda)s}} \frac{\partial z_k}{\partial x^{(\Lambda)t}} \times \frac{\partial z^i}{\partial \xi^e} \frac{\partial z^k}{\partial x^{(\Gamma)p}} g^{ae} g^{(\Gamma)pq} dV . \quad \textbf{(4)}$$

In these expressions a, b, c, d are indices of local element coordinate systems; $\overset{(e)}{\Omega}{}^{\Lambda}_{K}$ is the incidence matrix which maps global nodes into the local nodes of an element; P_K are the local values of shape functions; symbols $g^{(\Gamma)pq}$ are the components of the contravariant metric tensor at node Γ, indices p, q, r, s, t refer to the nodal coordinate system $x^{(\Lambda)p}$ at node Λ; i, j, k, l are the indices of global Cartesian coordinate system z^i of a model; symbols g^{ab} are the components of the contravariant metric tensor in the interior of an element and a, b, c, d are the indices of a local, element coordinate system ξ^a. Finally, A^{efgh} are the contravariant components of the elastic compliance tensor. Omission of the index e in (3), (4), etc., means global values of the quantities under consideration, i.e., simple summation of these.

In the Cartesian coordinates, applicable for flat two-dimensional and for three–dimensional configurations, expression (3) has the following interpretation:

$$\overset{(e)}{A}_{\Lambda st\Gamma uv} = \overset{(e)}{\Omega}{}^{L}_{\Lambda}\, \overset{(e)}{\Omega}{}^{M}_{\Gamma} \frac{\partial z^i}{\partial x^{(\Lambda)s}} \frac{\partial z^j}{\partial x^{(\Lambda)t}} \frac{\partial z^k}{\partial x^{(\Gamma)u}} \times \frac{\partial z^l}{\partial x^{(\Gamma)v}} \int_{\Omega_e} P_L A_{ijkl} P_M dV . \quad \textbf{(5)}$$

2.1 Matrix representation of the proposed technique

Herein one have to solve the system of equations (2). The first step will be the matrix representation of a problem. Matrix formulation will be made in the same manner as it has been done in [6]. Global system of equations (2) has the following matrix representation:

$$\left[A_{\Lambda st\Gamma uv}\right]\left\{t^{\Gamma uv}\right\} = \left\{B^{\Gamma q}_{\Lambda st} u_{\Gamma q}\right\}. \quad \textbf{(6)}$$

Matrix $\left[A_{\Lambda st\Gamma uv}\right]$ is positive definite, symmetric and sparse. At variance with underintegration which will be recommended for the computation of error indicators, this matrix (to avoid ill-conditioning) should be integrated by full Gaussian, or eventually Lobatto type [1] formulae.

2.2 Comparison of the FEDSS and some well known stress recovery methods

The numerical results of the present procedure are, up to the order of the rounding error, the same as in global stress smoothing methods proposed earlier by Oden [2], Hinton and Campbell [3], Zienkiewicz and Zhu [1], despite the fact that in the kernel of the system matrices (3) and (5) we have elastic coefficients, while in the aforementioned methods there are only the shape functions, $P_L P_M$, in the kernel. Of course, the right hand side is also different. Anyhow, such identical results are rather unexpected. It should be however pointed out that, to get identical results, the same set of the points of numerical integration should be taken for both methods. The advantage of the proposed procedure is a fact that it can be used also in the case when only the displacements are available, either from theoretical considerations, numerical analysis, or an experiment.

Also, special attention is paid on the tensorial invariance of the of corresponding expressions. Hence the present formulation allows use of different coordinate system at each observed node. Obviously, tensorial invariance can be kept independently on the kind of stress smoothing approach [8].

3 Determination of the strain energy and appropriate norms

Traditionally, the strain energy

$$U = \frac{1}{2}\int_{\Omega} \boldsymbol{t}^T \boldsymbol{A}\, \boldsymbol{t}\, d\Omega, \quad \textbf{(7)}$$

has been used [9] for estimation of accuracy and convergence of the finite element solutions. In this expression $\boldsymbol{t}$ is an exact stress solution, $\boldsymbol{A}$ the elastic compliance tensor and Ω the domain of the model. The energy of finite element solution at the element level is obviously:

$$U_h = \frac{1}{2}\int_{\Omega_e} \boldsymbol{t}_h{}^T \boldsymbol{A}\, \boldsymbol{t}_h\, d\Omega_e , \quad \textbf{(8)}$$

where Ω_e is the domain of an element. Note also that $\boldsymbol{t}_h$ is a finite element stress, dependent on the stress recovery procedure.

The popularity of above measure is partially due to a fact that it is, at a system level, because of the First Law of Thermodynamics, equal to the work of the external forces (at least for hyperelastic materials) which can be easily calculated [10].

However, if a local discretization error estimate for a given mesh is needed, it is practical to introduce so called energy (error) norm:

$$\|e\| = \left(\int_{\Omega} \left(\boldsymbol{t} - \boldsymbol{t}_h\right)^T \boldsymbol{A}\left(\boldsymbol{t} - \boldsymbol{t}_h\right) d\Omega \right)^{1/2} \quad \textbf{(9)}$$

It is usually taken as granted that smoothed (continuous) stress field is “more accurate” than finite element discontinuous stress pattern. However, the numerical tests show that the strain energy (8) at global level, if raw FE stresses are considered, is closer to (7) than if we use any sophisticated stress smoothing (see Figure 3). Hence, for the purpose of the stress estimation, in the equation (8) we can replace the exact stress $\boldsymbol{t}$ by $\boldsymbol{t}_{\mathrm{FEA}}$ – raw finite element results, and $\boldsymbol{t}_h$ by $\boldsymbol{t}_{smooth}$ – the result of any smoothing procedure.

The corresponding a posteriori error indicator of Z–Z (Zienkiewicz – Zhu) type [1] is, for an element:

$$\left\| e_{fea} \right\|_e = \left(\int_{\Omega_e} \left(\boldsymbol{t}_{fea} - \boldsymbol{t}_{smooth}\right)^T \boldsymbol{A} \left(\boldsymbol{t}_{fea} - \boldsymbol{t}_{smooth}\right) d\Omega_e \right)^{1/2} \quad \textbf{(10)}$$

where $\boldsymbol{t}_{\mathrm{FEA}}$ is a raw stress field from the displacement FE analysis, while $\boldsymbol{t}_{smooth}$ is a stress field obtained by some smoothing procedure. Analogously, one can introduce the absolute energy norm:

$$\|u\|_e = \sqrt{2U_h} \quad \textbf{(11)}$$

or more specifically:

$$\left\|u_{fea}\right\|_e = \left(\int_{\Omega_e} \boldsymbol{t}_{fea}^{\ T} \boldsymbol{A}\, \boldsymbol{t}_{fea} d\Omega_e \right)^{1/2} \quad \textbf{(12)}$$

It is easy now to define the *relative percentage error* [1] or *precision* [2] as:

$$\eta_e = \frac{\|e\|_e}{\|u\|_e} \times 100 \quad \textbf{(13)}$$

$$\left(\eta_{fea}\right)_e = \frac{\left\|e_{fea}\right\|_e}{\left\|u_{fea}\right\|_e} \times 100 \quad \textbf{(14)}$$

The element error indicator (14) is almost indentical to those proposed in [1] and [12]. Denominator in these references is defined as $\left(\|u\|_e^2 + \|e\|_e^2\right)^{1/2}$, assuring η_e <100% for coarse meshes, but approaching (11) for realistic ones.

However, most important difference between the present and previous approaches is *underintegration* (integration of $\|e\|$ in the derivative (stress) superconvergent points [13]) which decisively increases reliability of Z–Z type error estimators.

4 Numerical examples

In this section two types of numerical tests are performed. The first one is the study of quality of the proposed FEDSS stress recovery procedure. Second is an analysis of the problem of error estimation of finite element solution. In the available references, different stress recovery procedures are generally presented as superconvergent, more accurate than underlying finite element analysis. Numerical examples presented here will show that such results are probably obtained with integration formulae of higher order than it is justified and necessary when the energy error norm (indicator) is calculated.

4.1 One-dimensional example

Let us consider one – dimensional example proposed in [14]. For the purpose of this paper it is modelled by one row of two-dimensional elements. This model is particularly useful for the demonstration of different stress recovery approaches, like FEA, FEDSS and FEAavrg. Special attention is paid on the dependence of results on the order of the Gaussian quadrature. The exercise is sketched on Figure 1.

Modulus of elasticity and Poisson's coefficient are taken to be E = 1 and n = 0, respectively. The governing equations for this problem are:

$$t_{,x} - 4Ex = 0, \quad u(0) = 0, \quad u(3) = 52, \qquad t = E\frac{\partial u}{\partial x}. \quad \textbf{(15)}$$

Exact stress is:

$$t = E\left(2x^2 + \frac{34}{3}\right) \quad \textbf{(16)}$$

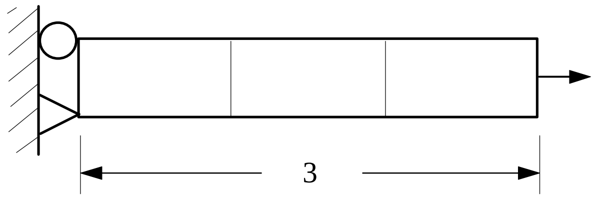

Figure 1. *One–dimensional example*

On Figure 2 the exact stresses, stress solutions of classical finite element analysis and two types of stress smoothing procedures, FEDSS and FEAavrg are shown.

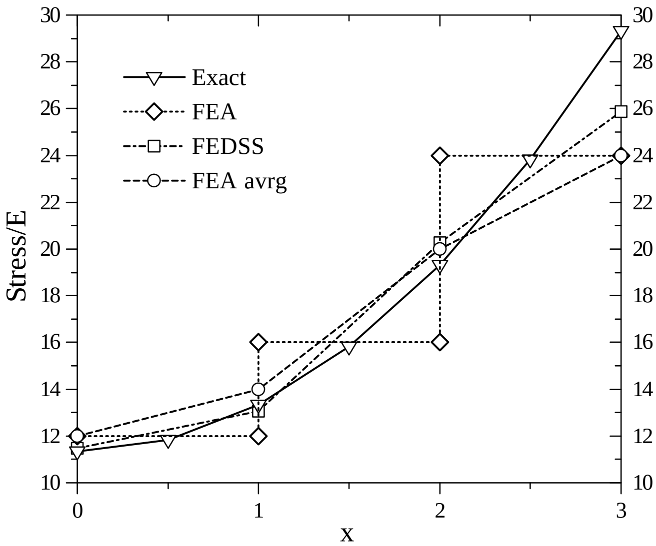

Figure 2. *Effects of the various stress recovery procedures*

The finite element stresses are discontinuous over the boundaries of elements, while exact and smoothed stresses are continuous. In the centres of elements the difference between EXACT and FEA stresses is smallest compared with the combinations EXACT – FEDSS and EXACT – FEAavrg. This is not the case for other positions of the Gaussian integration points where FEA solution is generally more in error than smoothed solutions. Consequently, numerical examinations show that for higher order Gaussian integration relative energy error norms are larger than for one point Gaussian quadrature, in contrary to the full strain energies, which are more accurate (closer to the exact values) when integrated by the higher order quadrature, Table 1.

Finally, one can conclude that the stress error, Figure 2, and the errors in energy, Table 1, smaller for the global (FEDSS) than for the local (FEAavrg) stress smoothing, are more clearly represented by one–point than by two–point Gaussian integration. In fact "accurate" integration of energy error norm integrates the square of a local gap (see Figure 2) between the exact and FE solution, and hence the energy error norm is inadequately represented by higher order integration.

method	strain energy exact=493.87	η 3 el	η 15 el
analytical 1×1	479.37	–	–
analytical 2×2	493.83	–	–
FEA 1×1	488.00	9.32	0.037
FEA 2×2	488.00	10.87	2.203
FEAavrg 1×1	471.00	7.97	0.826
FEA avrg 2×2	473.30	9.55	0.961
FEDSS 1×1	480.16	3.92	0.369
FEDSS 2×2	483.73	5.63	0.525

Table 1. *Strain energy and relative energy of one-dimensional model*

4.2 Rectangular in–plane loaded plate with the prescribed displacement

The problem is borrowed from [4]. Aim of this subsection is to show how the order of Gaussian quadrature influences the quality of stress recovery procedures in a two – dimensional mesh. Rectangular domain determined by the points (0,0), (2,0), (2,1) and (0,1) is considered. Modulus of elasticity is E = 1 and Poisson's coefficient n = 0. Analytical displacements are given by the relationship:

$$u = v = x\left(x-\frac{2}{3}\right)\left(x-\frac{3}{2}\right)(x-2)(1+y)+ \\ 10y\left(y-\frac{1}{3}\right)\left(y-\frac{3}{4}\right)(y-1)(1+x). \qquad \textbf{(17)}$$

Exact strains and stresses are calculated from the equations of elasticity. For the plate thickness 1, the total strain energy is $U = 5.29563$. Finite element models consist of bilinear rectangular four noded isoparametric elements. The sequence of meshes 2×2, 4×2, 4×4, 8×4, 8×8, 16×16, 32×32 is considered, with degrees of freedom 18, 30, 50, 90, 162, 578, 2178, respectively. Matrices in (6) were formed using 2×2 Gaussian quadrature.

The quantities of primary interest were the convergence of the *strain energy* and the *relative energy error norm* according to equations (7)–(14). Exact stresses are compared to stresses obtained with stress recovery methods FEA, FEDSS and FEAavrg. Stresses and expressions (7)–(14) were evaluated using 1×1, 2×2 and 3×3 Gaussian quadrature.

Strain energy convergence of the examined models is presented on Figure 3. From this Figure the following facts are evident:

- the exact strain energy, as a high order polynomial (1), is very much dependent on the order of the Gaussian integration rule,
- the strain energy of the raw finite element solution (FEA) is practically independent on integration rule, because it behaves more or less as a constant in the interior of each element,
- for the reasonably dense meshes FEA strain energy practically coincides with the theoretical value integrated at midpoints of elements,
- total strain energy for both FEDSS and FEAavrg models is less accurate than for raw FEA solution,
- both models are more sensitive to the order of integration rule than raw FE solution. However, the resulting differences are qualitatively insignificant,
- FEDSS approach is clearly more accurate than FEAavrg.

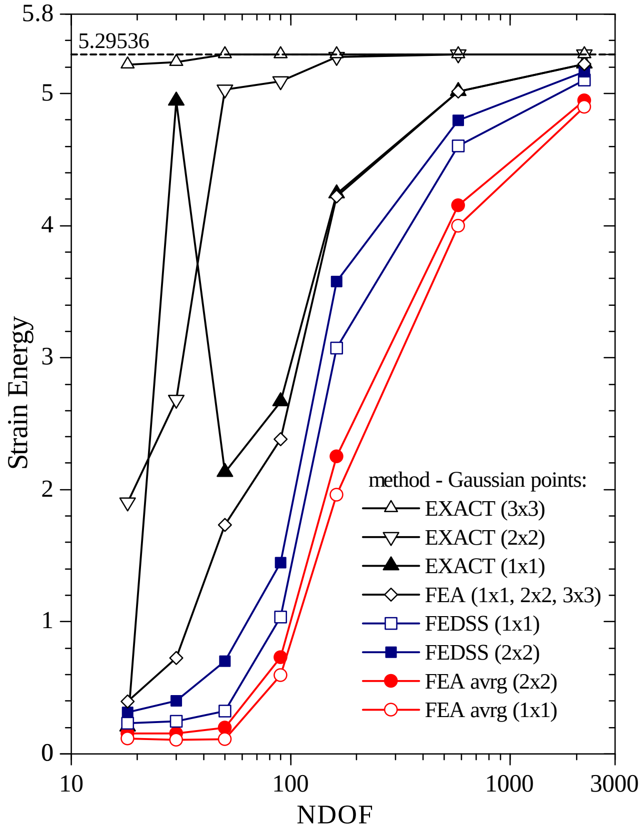

Figure 3. Strain energy convergence

Explanation is analogous to that for one–dimensional example, Figure 2, where at the element midpoint (superconvergent point [2, 13]) there is a smallest difference between FEA and exact stresses. Conclusion is that if a 1×1 Gaussian quadrature is used the exact solution can be replaced with FEA solution for linear or bilinear elements. It is expected that analogous conclusions can be made for the elements of higher order.

Figure 4 shows the relative energy error norm (with respect to the exact solution) (13) of different stress recovery solutions, versus the number of degrees of freedom. From this figure one can conclude that:

- the error norm of FEA solution is smallest when underintegrated (by 1×1 Gaussian quadrature). This is in the accordance with the fact that FEA and exact strain energy are very close (Figure 3) when 1×1 integration is employed,
- the error norms of FEAavrg and FEDSS are also smaller when integrated by 1×1 scheme than by 2×2 scheme, opposing formally the situation on Figure 3, where the integration of full strain energies is more accurate if 2×2 scheme is used, than 1×1. The reason for this discrepancy is a fact than smoothed solutions (see Figure 2) are similarly to the generic FEA solution, closest to the exact solution at the element midpoints (superconvergent points).

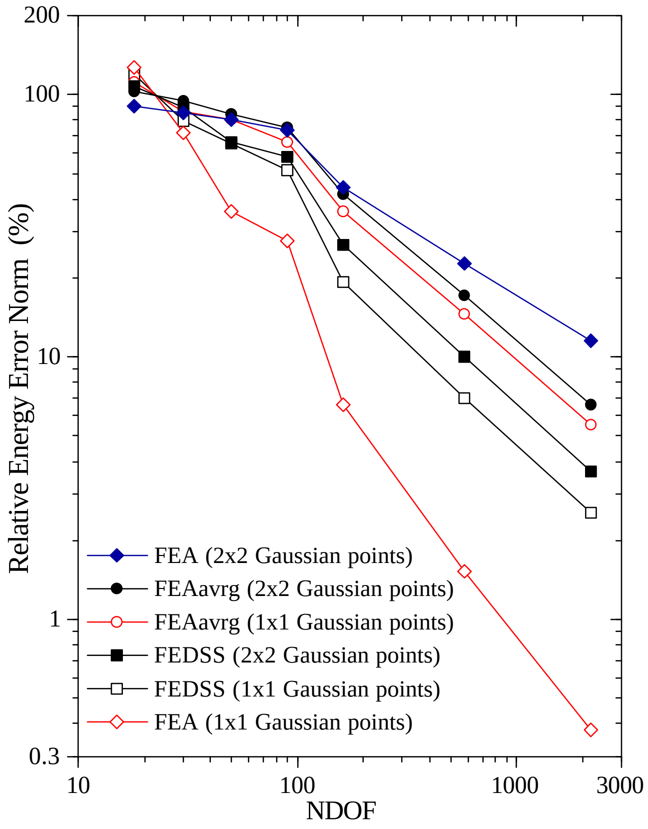

Figure 4. *Relative energy error norm of the stresses with respect to the exact values*

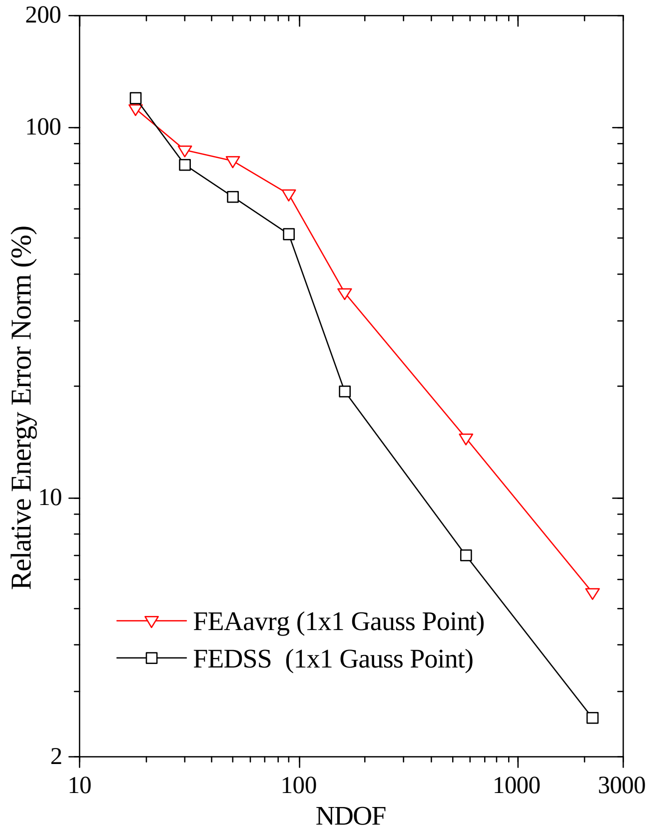

Figure 5. *Relative energy error norm of the stresses related to the exact values*

However, on both Figures 3 and 4, FEDSS and FEAavrg curves are qualitatively similar, inrespectively on the integration scheme used. What is more important from both figures one can conclude, without any doubt, that FEDSS is more accurate smoothing scheme than classical FEAavrg.

In the Figure 5 FEAavrg and FEDSS errors with respect to the theoretical values, integrated by 1×1 Gaussian quadrature, extracted from the Figure 4, are presented.

In Figure 6 the same values integrated by 2×2 quadrature are given. Similarly in Figure 7 errors with respect to FEA values (i.e. Z-Z type error indicators) integrated by 1×1 rule are given.

Finally, in Figure 8 error indicators integrated by full, 2×2 rule are shown. It is evident that for reasonably dense meshes the character of the curves on Figure 5 and Figure 7 is very similar, both qualitatively and quantitatively. In contrary, comparison of Figures 6 and 8 show that usual, full, 2×2 integration of the error indicator, masks the difference between FEAavrg and FEDSS procedures.

Shortly speaking, the strain energy of the FEDSS (and any other equivalent or similar procedure) is more accurate compared with, say, that of simple stress averaging. However, this fact is dubious if for the calculation of Z-Z type error indicator one uses “exact”, 2×2 in the case considered, integration. At variance, the advantage of more advanced procedures is clear if the derivative superconvergent points (in the present case 1×1) are used, see Tables 2 and 3.

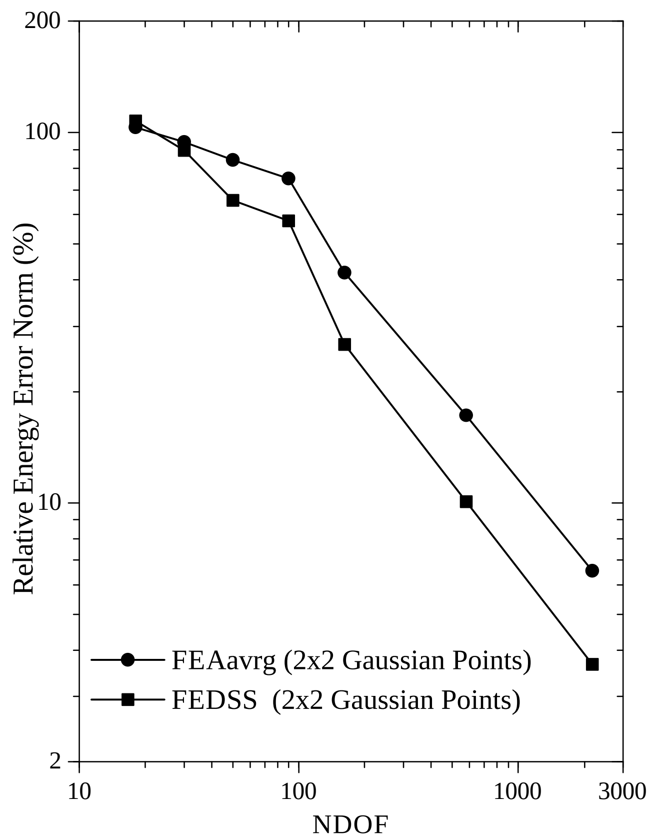

Figure 6. *Relative energy error norm of the stresses related to the exact values*

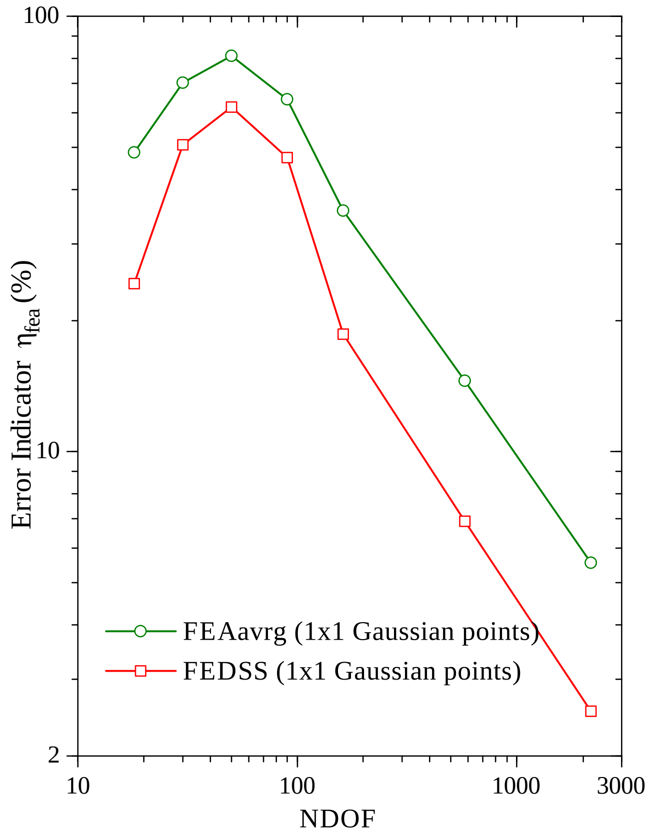

Figure 7. *Error indicators integrated at the element midpoint*

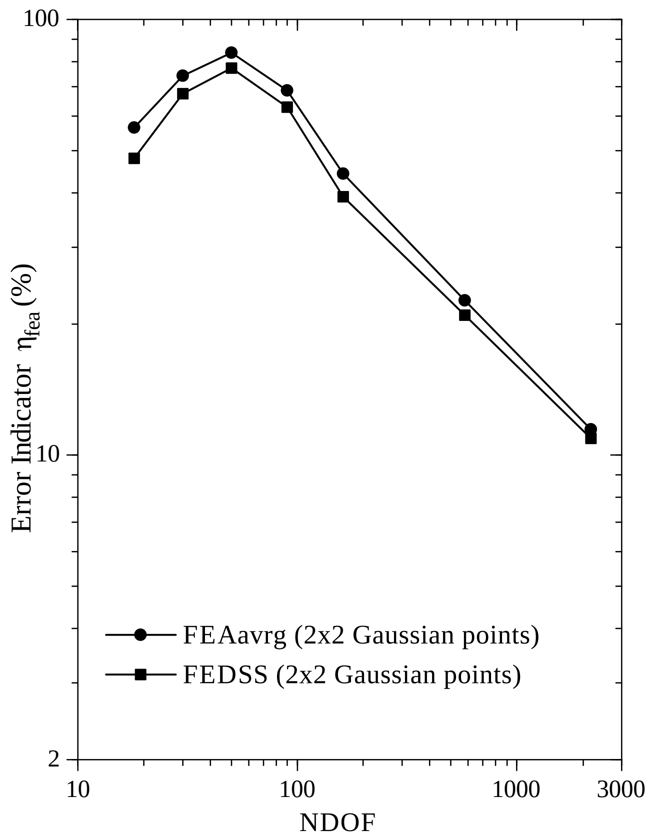

Figure 8. *Error indicators integrated by the full Gaussian quadrature*

NDOF	η^{FEA}	η^{FEDSS}	$\eta^{FEAavrg}$
18	127.73	119.44	111.57
30	71.44	79.38	86.59
50	35.91	65.08	80.77
90	27.56	51.37	66.06
162	6.59	19.30	35.83
578	1.52	6.98	14.51
2178	0.38	2.55	5.55

Table 2. *Relative energy norms (%), 1´1 integration*

NDOF	η^{exact}_{FEA}	η^{FEDSS}_{FEA}	$\eta^{FEAavrg}_{FEA}$
18	290.35	24.34	48.67
30	186.07	50.63	70.34
50	39.80	61.89	84.13
90	29.45	47.31	64.46
162	6.60	18.65	35.80
578	1.52	6.91	14.53
2178	0.38	2.53	5.56

Table 3. *Error indicators (%),1´1 in tegration*

4.3 Square plate with a circular hole

A problem of the square plate with a circular hole [15] is considered in Figure 9. Only a quarter of it is analyzed due to the symmetry of that system. Isotropic, homogeneous material properties and the plane stress behavior are assumed. Modulus of elasticity is 1 and Poisson ratio 0.3. Plain isoparametric four-noded quadrilateral elements and 2×2 Gauss quadrature are used.

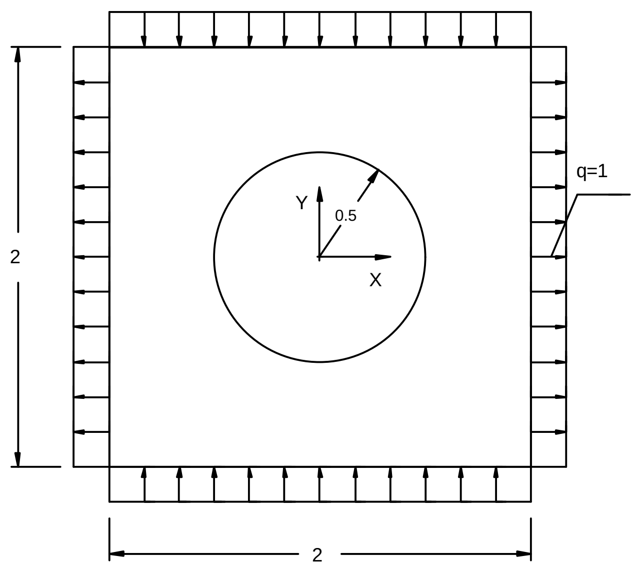

Figure 9. *Plate with a circular hole*

Recovered FEDSS stresses are compared with simple nodal averaged stresses–FEAavrg. To study the effectiveness of FEDSS method as a stress recovery and smoothing procedure two types of the analysis were performed. First, Z–Z type error indicator versus number of elements, and second, the same indicator as a function of the execution time, is examined.

The numerical studies were made for the sequence of meshes from 1×1 to 30×30. One of them (5×5) is presented in Figure 10.

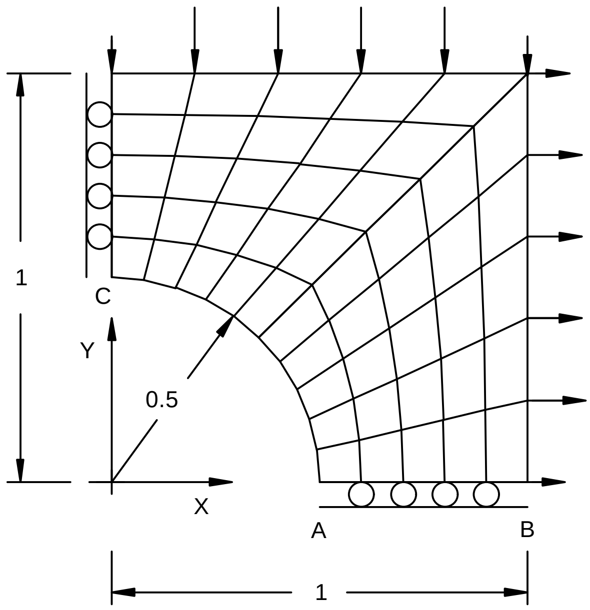

Figure 10. *Finite element mesh*

One possible user defined set of coordinate systems, useful for the determination of hoop stresses at global nodes is shown on Figure 11.

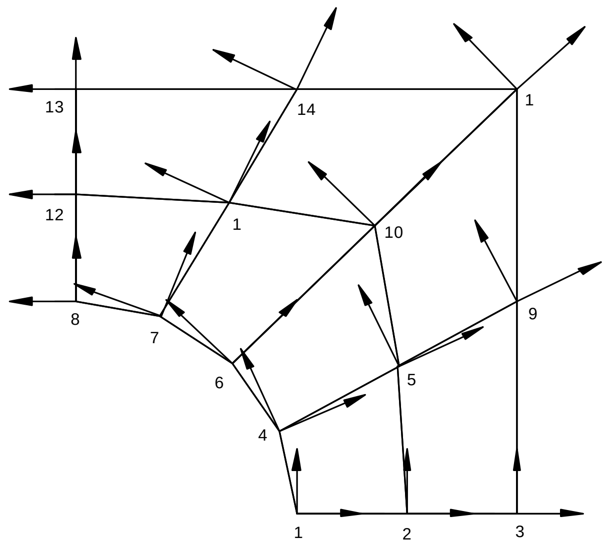

Figure 11. *User defined coordinate systems.*

Figure 12 shows a comparison of the error indicators versus the number of degrees of freedom. As it can be seen, FEDSS formulation, presented here, is significantly more accurate than conventional smoothing FEAavrg.

Even more important is a quantitative measure, i.e., the execution time for the accuracy required. In Figure 13 error indicators related to the time of execution are compared.

It can be said that FEDSS stress recovery procedure evidently increases the quality of FE solution and reduces the solution time. The error of 5% measured in the energy error norm by the use of 1×1 integration rule is achieved twice faster by the method presented in this paper, compared with classical averaging procedure, despite the additional time necessary for the solution of a system (6) for stresses.

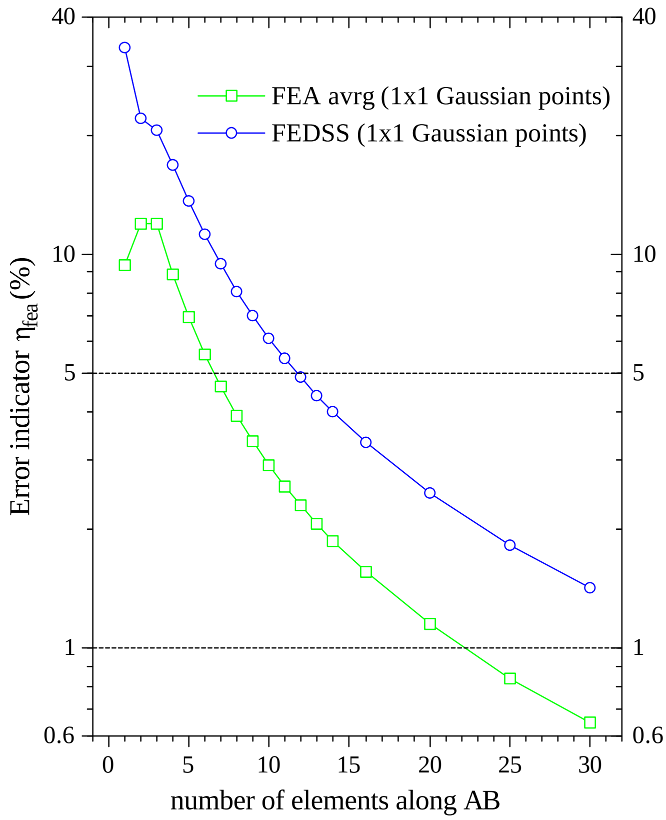

Figure 12. *Error indicators versus mesh density*

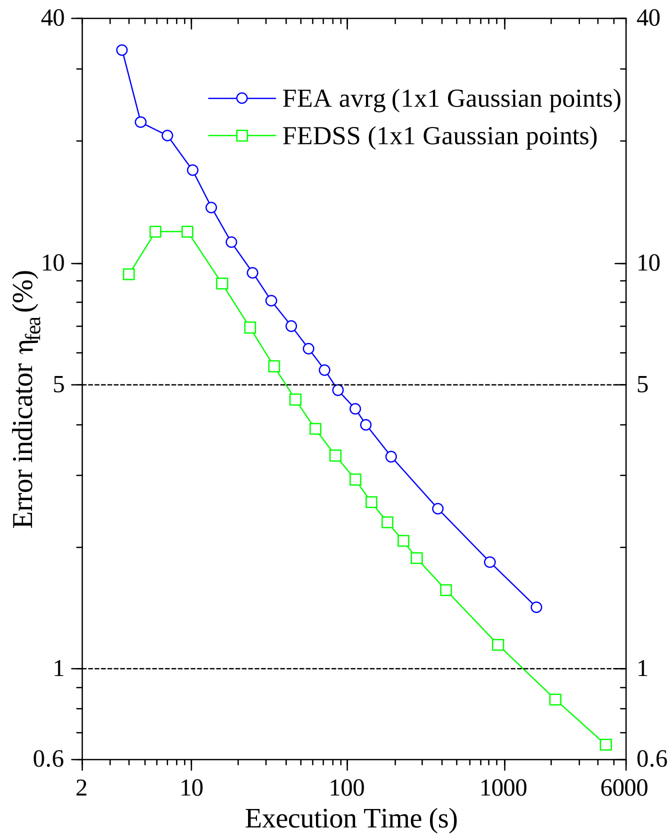

Figure 13. *Error indicators versus execution time*

Note that quantitative results (accuracy in the function of the solution time) can seldom be found in the available references on the postprocessing procedures.

5 Conclusions

New global, coordinate independent approximation of continuous displacement field has been presented and applied to two-dimensional elasticity problems. Proposed procedure delivers the needed accuracy in a significantly shorter time than underlying classical finite element analysis. However, as it has been noted in Subsection 2.2, many of the global interpolation procedures are of the similar accuracy. But, the advantage of the present algorithm is its flexibility (i.e. possibility to use arbirtrary local coordinate systems). In addition, it is implemented as an universal postprocessing routine for which the input data (displacements) can be taken from any finite element analysis package, theory, or experiment.

However, the most important new detail in the above study is perhaps a fact that underintegration (integration in the derivative (stress) superconvergent points) gives much sharper picture of the error indicator. This fact recommends the use of the proposed approach not only for the quality assessment of the postprocessing procedures, but also as a precise tool in the adaptive mesh refinement algorithms.

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