

ON THE EFFICIENCY OF THE STRESS PROJECTION PROCEDURE

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Abstract. In the present paper the computer time execution efficiency of the FEGSS (Finite Element Global Stress Smoothing) procedure is examined. After testing of the present global method on several numerical examples it can be concluded that global projection is definitively superior, at least compared with local stress averaging, also with respect to the computer execution time for the accuracy required. Furthermore, when the model problem is a three-dimensional or flat two-dimensional one, the global FEGSS method can be re-established in the disjoint form and in this particular, but very common case, we have stress recovery procedure an order of magnitude more efficient than global FEGSS or simple averaging.

1. Introduction. From the very beginning of the development and introduction of Finite Element (FE) method as one of the basic tools for calculating the response and the performance of structures, obtaining of the accurate and unambiguous (state of stress) stress picture over the model is a highest priority task. Until now, because of the easy computer implementation, relatively simple final expressions and low computer time consumption, the most popular way of calculating the behavior of the structure has been via h -version of the displacement finite element approach with the additional use of constitutive relations for calculating raw FE stresses.

The main disadvantage of the FE displacement approach (based on the theorem of minimum potential energy) is that calculated stresses are generally discontinuous at the element interfaces. It simply means that, in general case, there is no unique value of the stress in the global node, but as many different stress values as there is a elements common to that node. Further, use of low order quadrilateral elements results with the low order discontinuous approximation. As a consequence there is a nonuniform stress accuracy from point to point within a element. Even more, in the places of the stress concentration there is a false picture that even with the coarse meshes we have accurate enough stress state according to theoretical value, but after careful and extensive numerical examination it can be concluded that this is simply due the oscillatory nature of the finite element stresses calculated from FE displacements with the weak form of Hookean law.

As an attempt to overcome the problem of interpreting the results of numerically discontinuous model of a physically continuous system and improve the overall stress results, the number of different techniques were introduced.

2. Recapitulation of stress recovery techniques. As a one of the earliest attempt to obtain the smoothed stress picture of the model was the averaging of the nodal stresses of all elements meeting at a common node (in the present paper FEAavg). This is simple and fast procedure, but there is a numerous examples where cannot be adopted, see [1]. Nevertheless, from nowadays point of view it is classical technique and widely adopted as a reference procedure in a numerical examinations.

Further, procedure so called "consistent conjugate stress calculation" [2] was introduced in 1971. This method "...is based on the idea of consistent stress approximation and it approximates such stresses using the notion of a domain influence of the stress intensity at a nodal point". This is *global* stress smoothing method result in set of linear simultaneous equations which matrix is well conditioned and positive definite.

Also, procedures named local (with or without reduced integration) and global stress smoothing procedures, based on least squares smoothing of the unsmoothed data [1] were presented in 1974. *Local* smoothing procedure is applied to each element separately and hence it does not produce smooth stress state of the model, so it is rather stress recovery procedure than stress smoothing procedure. Local smoothing may be interpreted as a local

interpolation/extrapolation scheme using a bilinear surface $g(\xi, \eta) = a\xi + b\eta + c\xi\eta + d$ and the stress values at Gauss points. On the other hand, the *global* smoothing procedure is carried out over the whole finite element model, where element smoothing matrices and force vectors are assembled into the overall smoothing matrix and force vector. Problem is stated in the way of finding the coefficients a_{ij} which minimize the functional

$$\chi = \iint (\sigma - g)^2 dx dy. \quad (1)$$

In order to χ to be minimum there are a set of linear simultaneous equations in a_{ij} known as the *normal* equations:

$$\frac{\partial \chi}{\partial a_{ij}} = 0. \quad (2)$$

The authors says that despite “...this method produced well conditioned, positive definite matrices – it was found to be costly (a re-analysis is necessary) and was not consistently better than nodal averaging.” It should be noted that this approach is very similar to method of “conjugate stress approximations” outlined above.

One of the most popular way of obtaining an improved solution is Superconvergent Patch Recovery (SPR) method [3] first appeared in 1987. The SPR is a least square fit of derivatives at superconvergent points. Since SPR solutions do not satisfied the governing equations or equilibrium equations, an improvement was introduced so that equilibrium equations are satisfied in a least square sense and this approach is named Superconvergent Patch Recovery using Equilibrium conditions (SPRE). Using an usual, but under question (see Chapter 3.), error analysis, where is assumed that smoothed solution is a always better approximation than raw FEA solution, authors were concluded that these procedures, which are based on local patches of elements, provides *superconvergent* derivatives over patches of elements. But, both approaches perform better in the interior domain than on patches near boundaries. Assuming that this behavior is due to reason that these methods do not intend to satisfy the boundary conditions, the SPREB approach is introduced in which the boundary conditions are considered also.

Without going in further enumeration, it should be noted that the question of time efficiency has been, as a rule, neglected in the validation process of stress recovery techniques. There is a broaden opinion that global projection methods increases accuracy, but even more the execution time. The present authors recently revisited the problem, and reported satisfactory efficiency of the global stress recovery procedures on the basis of the energy and supreme norms [4, 5].

2. Basic equations In this section the FEGSS global stress smoothing procedure first announced in [6,7] will be shortly represented. This procedure is a generalisation of the method proposed in [3] as a error estimation technique, but enriched with property of coordinate independence, according to tensorial character of finite element expressions first suggested in [8]

Let C_h be the finite element partitioning of the domain of the model problem $\bar{\Omega}$ into the assembly of finite elements $K = \{\Omega_e, e = 1, \dots, N\}$. Because we already know discontinuous stresses σ_h , it is customary to project them onto some suitably chosen finite element space T_h consisting of continuous functions. This can be done by letting: $T_h = \{s \in C(\bar{\Omega}) \mid s|_K \in R_K(K), \forall K \in C_K\}$, where R_K are the interpolation functions, and then defining the smoothed stresses $\sigma_h^* \in [T_h]^{n \times n}$ through:

$$(\sigma_h^*, s) = (\sigma_h, s), \quad \forall s \in [T_h]^{n \times n}. \quad (3)$$

A weak solution of (3) one can get by the Galerkin procedure using stress variations as the test functions. Finite element approximations are introduced similarly as in [9]. After recasting of all quantities in the coordinate form, one obtains the relationship:

$$V_{\Lambda st} \Gamma_{uv} \sigma^{\Gamma uv} = P_{\Lambda st} \quad (4)$$

where, for the domain Ω_e of an element e :

$$V_{\Lambda st \Gamma uv}^{(e)} = \Omega_{\Lambda}^L \Omega_{\Gamma}^M \int_{\Omega_e} P_L \frac{\partial z_i}{\partial x^{(\Lambda)s}} \frac{\partial z_k}{\partial x^{(\Lambda)t}} \frac{\partial z^k}{\partial x^{(\Gamma)u}} \frac{\partial z^i}{\partial x^{(\Gamma)v}} P_M d\Omega_e, \quad (5)$$

$$P_{\Lambda st}^{(e)} = \Omega_{\Lambda}^L \int_{\Omega_e} P_L \frac{\partial z_i}{\partial x^{(\Lambda)s}} \frac{\partial z_k}{\partial x^{(\Lambda)t}} P_K \frac{\partial z^i}{\partial \xi^{(\kappa)\alpha}} \frac{\partial z^k}{\partial \xi^{(\kappa)\beta}} \sigma^{\kappa\alpha\beta} d\Omega_e. \quad (6)$$

In these expressions Ω_K^{Λ} is the incidence matrix which maps global nodes into the local nodes of an element; P_K are the local values of shape functions; symbols $g^{(\Gamma)pq}$ are the components of the contravariant metric tensor at node Γ ; indices s, t, u, v refer to the nodal coordinate system $x^{(\Lambda)p}$ at node Λ ; i, k are the indices of global Cartesian coordinate system z^i of a model; a, b are the indices of a local, element coordinate system ξ^a . Omission of the index e in (5), (6), etc., means global values of the quantities under consideration, i.e., simple summation of these.

In the Cartesian coordinates, applicable for flat two-dimensional and for three-dimensional configurations, where coordinate systems in all global nodes are parallel with global coordinate system of a model, expressions (5) and (6) have the following interpretation:

$$V_{\Lambda st \Gamma uv}^{(e)} = \Omega_{\Lambda}^L \Omega_{\Gamma}^M \int_{\Omega_e} P_L \delta_{sv} \delta_{tu} P_M d\Omega_e, \quad (7)$$

$$P_{\Lambda st}^{(e)} = \Omega_{\Lambda}^L \int_{\Omega_e} P_L P_K \delta_{is} \delta_{kt} \sigma^{Kik} d\Omega_e. \quad (8)$$

In this case global FEGSS method can be re-established in the disjoint form allowing separate calculation of each stress component at the time, in the present paper FEGSS&ECS (Finite Element Global Stress Smoothing of Each Component of stress Separately).

In addition, formulation (4) allows introduction of the stress boundary conditions as the essential ones, in the present paper FEGSS&BC.

2. Error analysis There are several ways of investigation of the FEA error. Commonly used way of determination of an approximation error is by the use of an appropriate norm. If the discretization error estimate for a given mesh is needed, it is practical to introduce energy (error) norm:

$$\|e\| = \left(\int_{\Omega} (\mathbf{t} - \mathbf{t}_h) : \mathbf{A} : (\mathbf{t} - \mathbf{t}_h) d\Omega \right)^{1/2}. \quad (9)$$

In this expression \mathbf{t} is an exact stress solution, \mathbf{t}_h finite element stress, \mathbf{A} the elastic compliance tensor and Ω the domain of the model. Also, the *strain energy*:

$$U = \frac{1}{2} \int_{\Omega} \mathbf{t} : \mathbf{A} : \mathbf{t} d\Omega, \quad (10)$$

has been traditionally used [10] for estimation of accuracy and convergence of the finite element solution. The popularity of above measure is partially due to a fact that it is, at a system level, because of the First Law of Thermodynamics, equal to the work of the external forces (at least for hyperelastic materials).

Now, one can introduce the total energy norm:

$$\|U\| = \sqrt{2U}, \quad (11)$$

It has been proven in [11] that the energy of the error of the finite element solution is equal to the error of the energy, i.e.

$$\|e\|^2 = |2U - 2U_{\text{FEA}}|. \quad (12)$$

As simple differences between two scalar quantities, expressions of the type (14) can be used for the determination of the energy difference between any two solutions of the problem under consideration. At variance, the expressions of the type (9) calculate energy error correctly only if the stresses under consideration are based on some global projection procedure. The use of (9) in the case when t_h is *not* an orthogonal projection of t , can seriously underestimate the energy error norm. To be perfectly clear, note that use of (9) for the calculation of the energy error is justified if and only if $t_h = t_{\text{FEA}}$. Hence, the popular use of (9) for the determination of the error of a smoothed solution ($t_h = t_{\text{SMOOTH}}$) is overoptimistic and misleading.

It is easy now to define the *relative percentage error* [1] or *precision* [11] as:

$$\eta = 100 \frac{\|e\|}{\|U\|}, \quad (13)$$

When a largest local error is required, the supreme norm is usually applied. Per instance,

$$\|e_{\text{SMOOTH}}\|_{\infty} = \max_L |t_L^{jk} - t_{L\text{SMOOTH}}^{jk}| \quad (14)$$

is a largest stress error of the smoothed stress component t^{jk} , obtained at node L .

3. Numerical examples In this section the time efficiency of the proposed FEGSS stress smoothing technique is illustrated by means of a plane stress numerical example. The problem of a square plate with a circular hole [11] is shown on Figure 1. Isotropic, homogeneous material properties and the plane stress behavior are assumed. Modulus of elasticity is 1 and Poisson ratio 0.3. Plain isoparametric four-noded quadrilateral elements and 2×2 Gauss quadrature are used.

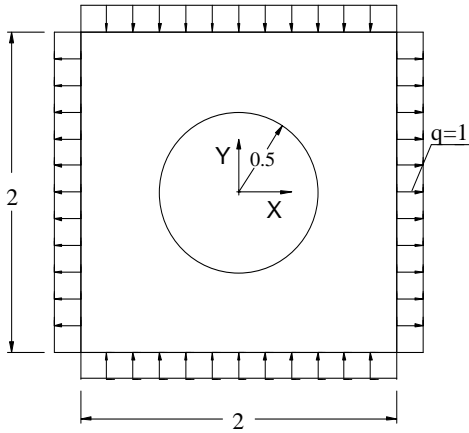


Figure 1. Plate with a circular hole

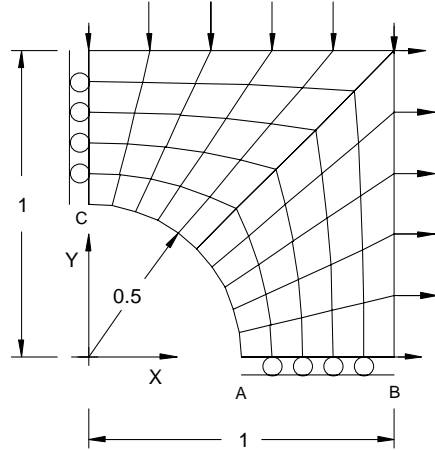


Figure 2. Finite element mesh

Recovered FEGSS stresses are compared with simple nodal averaged stresses—FEAavg. To study the effectiveness of FEDSS method as a stress recovery and smoothing procedure two types of analysis were performed. First, Z-Z type error indicator versus number of elements, and second, the same indicator as a function of the execution time, is examined.

4. Conclusion After testing of the present global FEGSS method on several numerical examples we can conclude that despite the additional time needed for solving the resulting system of linear equations, proposed procedure evidently increases the quality of FE solution and, for finer meshes, can even reduce the total solution time, at least compared with the simplest form of stress averaging, i.e. the arithmetic mean of nodal stresses from neighbouring elements [11]. In addition, when our model problem is a three-dimensional or flat two-dimensional one, Cartesian coordinates can be used for either global and local references, and the global FEGSS method can be reestablished in the disjoint form FEGSS&ECS, allowing separate calculation of each stress component. There is a numerical evidence that in this particular, but very common case, we have stress recovery procedure an order of magnitude more efficient than global FEGSS or simple averaging.

Finally, it can be said that global projection is definitively superior, at least compared with local stress averaging, also with respect to the computer execution time for the accuracy required.

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