Finite Element Simulations in Elastic and Heat Transfer Analysis

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In the present paper, execution times of direct MA47 procedure for solving large scale linear systems of equations, arising from the primal-mixed finite element approximation procedure in heat transfer and elastic analysis in solid mechanics, obtained on two different computer platforms: NEC SX6i 8GbRAM and PC Configuration Pentium IV 2.4 GHz, 1GB RAM, SCSI HDD 2x36GB, are compared and/or reported. It will be shown that NEC supercomputer enable greater design and development freedom, through multiple minimisation of the computer time needed per prescribed accuracy, as well enabling that mechanical system with much more degrees of freedom are solved, which is one of the key steps in proving the stability of any new numerical procedure based on the Galerkin finite element method.

Key Words: Solid Mechanics, Finite Element, Mixed Formulation, Large Scale, Sparse

FORMULATION

In the present paper the novel techniques in the semicoupled elastostatics [1] and transient heat or steady state heat transfer analysis [2] of conventional materials, are considered. Both of the considered analysis, elastostatics and steady state heat, are derived from the corresponding mixed formulations, where local base functions for approximation of primal and dual variables (displacement/stress and temperature/heat flux) are chosen from continuous finite element sub-spaces. These equations are one-way coupled in the sense that the field equation for T does not involve $\boldsymbol{u}$ but that for $\boldsymbol{u}$ involves T. Thus the temperature field can be found first, and then displacements can be computed.

The essential contribution of this approach is that dual variables, stresses or heat fluxes, are presently fundamental variables, simultaneously calculated with primal ones, i.e. displacements and heat fluxes, respectively. In addition, present approach is fully three-dimensional, and every body is considered as geometrically three-dimensional, regardless of its thickness. The main contribution of such approach is in overcoming of well-known transition problem of connecting finite elements of different types and dimensionality (e.g. shell to solid). Moreover, for the reason that full fourth-order tensor of elastic compliance and second order tensor of thermal conductivity is used, we may analyze arbitrary isotropic and orthotropic materials.

FINITE ELEMENT EQUATIONS IN ELASTOSTATICS

It has been shown in [1] that primal-mixed approach in elastostatics based on Hellinger-Reissner principle, can be written as the system of linear equations of order $n = n_u + n_t$, where n_u is the number of displacement degrees of freedom, while n_u is the number of stress degrees of freedom:

$$\begin{bmatrix} \mathbf{A}_{vv} & -\mathbf{D}_{vv} \\ -\mathbf{D}_{vv}{}^{T} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{t}_v \\ \mathbf{u}_v \end{bmatrix} = \begin{bmatrix} \mathbf{-A}_{vp} & \mathbf{D}_{vp} \\ \mathbf{D}_{pv}{}^{T} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{t}_p \\ \mathbf{u}_p \end{bmatrix} - \begin{bmatrix} 0 \\ \mathbf{f}_p + \mathbf{p}_p \end{bmatrix}, \tag{1}$$

where the members of the matrices $\mathbf{A}$ and $\mathbf{D}$, of the column matrices $\mathbf{f}$ and $\mathbf{p}$ (discretized body and surface forces) in (1), are respectively:

$$A_{\Lambda uv\Gamma st} = \sum_{e} \int_{\Omega_i} \Omega_{\Lambda}^{N} S_N \, g^{\,a}_{(\Lambda)u} g^{\,b}_{(\Lambda)v} A_{abcd} \, g^{\,c}_{(\Gamma)s} g^{\,d}_{(\Gamma)t} T_L \Omega_{\Gamma}^{L} d\Omega, \tag{2}$$

$$D^{\Gamma q}_{\Lambda uv} = \sum_{e} \int_{\Omega_i} \Omega_{\Lambda}^{N} S_N U_a^K \Omega_K^{\Gamma} g^{\,a}_{(\Lambda)u} \, g^{(\Gamma)q}_{(\Lambda)v} d\Omega, \tag{3}$$

$$f^{\Lambda q} = \sum_{e} \int_{\Omega_i} g_a^{(\Lambda)q} \Omega_M^{\Lambda} V^M f^a d\Omega, \tag{4}$$

$$p^{\Lambda q} = \sum_{e} \int_{\partial\Omega_{it}} g_a^{(\Lambda)q} \Omega_M^{\Lambda} V^M p^a d\partial\Omega. \tag{5}$$

In the above formulation, $\boldsymbol{u}$ is the displacement field, $\boldsymbol{T}$ is the stress field, $\boldsymbol{f}$ is the vector of body forces and $\boldsymbol{p}$ is the vector of boundary tractions, $\boldsymbol{w}$ is the vector of prescribed displacements, while $\boldsymbol{A}$ is fourth order compliance tensor. Further, $\Omega \subset R^n,\ n = 1,2,3$ in an open bounded domain of the elastic body, where n is the number of spatial dimensions considered. Hence, $\boldsymbol{n}$ is the unit normal vector to the boundary $\partial\Omega$, while $\partial\Omega_u$ and $\partial\Omega_t$ are the portions of $\partial\Omega$ where the displacements or stresses are prescribed, respectively. The $\{\boldsymbol{u}, \boldsymbol{T}\}$ and $\{\boldsymbol{v}, \boldsymbol{S}\}$ are pairs of trial and test displacement and stress functions, respectively, chosen from space , where H^1 which is the space of all scalar fields which are square integrable and have square integrable gradients, with the norm $\|g\|^2 = \int_{\Omega} \left((g^{'})^2 + g^2 \right) d\Omega$ for all $g \in H^1(\Omega)$.

Interested reader may found further information in [1], where it is shown that present approach is reliable [1], namely, it satisfies all convergence requirements.

Let's say something of the introduction of the calculated thermal filed in the present elastostatics approach. In the case of traditional materials where there is no heat production due to strain rate, thermal effects on a body are limited to strains due to the temperature gradient, which are autonomously determined and constitute only a datum for stress analysis. Therefore, in the present case, when we determine temperature field from finite element system equations (7), it is possible to calculate corresponding thermomechanical behavior from system (1).

Recall that strain filed $\boldsymbol{e}$ due to the temperature is given by:

$$\boldsymbol{e} = \alpha \left(T - T_o \right), \tag{6}$$

where α is second order tensor of thermal expansion coefficients, which in the case of isotropic material reduce to one constant value, or a spherical tensor if the material is orthotropic.

FINITE ELEMENT EQUATIONS IN TRANSIENT HEAT ANALYSIS

By analogy with finite element approach in elasticity [1], after discretization of the starting problem using finite element method, present scheme can be written [2] as a system of linear equations of order $n = n_q + n_T$, where n_T is the number of temperature degrees of freedom, while n_q is the number of flux degrees of freedom, in matrix form at the current time t:

$$\begin{bmatrix} \mathrm{A}_{vv} & \mathrm{B}_{vv}{}^{T} \\ \mathrm{B}_{vv} & -\mathrm{D}_{vv}-\mathrm{S}_{vv} \end{bmatrix} \begin{bmatrix} \mathbf{q}_v \\ \mathrm{T}_v \end{bmatrix} = \begin{bmatrix} -\mathrm{A}_{vp} & -\mathrm{B}_{vp}{}^{T} \\ -\mathrm{B}_{vp} & \mathrm{D}_{vp} \end{bmatrix} \begin{bmatrix} \mathbf{q}_p \\ \mathrm{T}_p \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{S}_{vp} \end{bmatrix} \begin{bmatrix} 0 \\ \mathrm{T}_p^{(t-1)} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathrm{F}_p + \mathrm{H}_p - \mathrm{K}_p \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathrm{L}_p^{(t-1)} \end{bmatrix} \quad (7)$$

In this expression, unknown (variable) and known (initial, prescribed) values of the fluxes and temperatures, denoted by the indices v and p respectively, are decomposed.

The nodal flux (q^{pL}) and temperature (T^L) components are consecutively ordered in the column matrices $\mathbf{q}$ and T respectively. The homogeneous and nonhomogenous essential boundary conditions per temperature T_p and heat flux $\boldsymbol{q}_p$ are introduced as contribution to the right-hand side of the expression (7).

The members of the entry matrices A, B, D and S, and the column matrices F, H, K and L in (7), are respectively:

$$\begin{aligned}
&A_{\Lambda p\Gamma r} = \sum_e \int_{\Omega_e} \overset{(e)}{\Omega}{}^L_\Lambda\, g^a_{(L)p} V_L k^{-1}_{ab} g^b_{(M)r} V_M\, \overset{(e)}{\Omega}{}^M_\Gamma\, d\Omega_e; \quad B_{\Lambda p\Gamma} = \sum_e \int_{\Omega_e} \overset{(e)}{\Omega}{}^L_\Lambda\, g^a_{(L)p} V_L P_{M,a}\, \overset{(e)}{\Omega}{}^M_\Gamma\, d\Omega_e \\
&D_{\Lambda\Gamma} = \sum_e \int_{\partial\Omega_{ce}} h_c\, \overset{(e)}{\Omega}{}^L_\Lambda\, P_L P_M\, \overset{(e)}{\Omega}{}^M_\Gamma\, \partial\Omega_{ce}; \quad F_\Gamma = \sum_e \int_{\Omega_e} \overset{(e)}{\Omega}{}^M_\Gamma\, P_M\, f d\Omega_e \\
&H_\Gamma = \sum_e \int_{\partial\Omega_{he}} \overset{(e)}{\Omega}{}^M_\Gamma\, P_M h d\partial\Omega_{he}; \quad K_\Gamma = \sum_e \int_{\partial\Omega_{ce}} \overset{(e)}{\Omega}{}^M_\Gamma\, P_M h_c T_0 d\partial\Omega_{ce} \\
&S_{\Lambda\Gamma} = \sum_e \int_{\Omega_e} \frac{\rho c}{\Delta t} \overset{(e)}{\Omega}{}^L_\Lambda\, P_L P_M\, \overset{(e)}{\Omega}{}^M_\Gamma\, d\Omega_e; \quad L_\Gamma = \sum_e \int_{\Omega_e} \frac{\rho c}{\Delta t}\, {}^{n-1}T_{(M)}\, P_M\, \overset{(e)}{\Omega}{}^M_\Gamma\, d\Omega_e
\end{aligned} \quad (8)$$

The above expression is evaluated for each unsuppressed degree of freedom of the heat flux vector or temperature, connected to the global node Λ and/or Γ of the considered finite element mesh, where $\overset{(e)}{\Omega}{}^L_\Lambda$ is a connectivity operator, which maps the set of global nodes Λ into the set of local nodes L per elements, and vice versa. The Euclidian shifting operator $g^a_{(L)p}$ is given by $g^a_{(L)p} = \delta_{ij} g^{ac} \left(\partial z^i / \partial \xi^c\right)\left(\partial z^j / \partial y^{(L)p}\right)$, where, $z^i\,(i,j,k,l = 1,2,3)$ is global Cartesian coordinate system of reference. Further, $y^{(L)r}\,(r,s,t = 1,2,3)$ is coordinate system at each global node L, per heat flux. Further, local natural (convective) coordinate systems per finite elements are denoted by $\xi^a\,(a,b,c,d = 1,2,3)$, while, g^{ab} and $g^{(L)mn}$ are components of the contravariant fundamental metric tensors, the first one with respect to natural coordinate system of a finite element ξ^a, and the second to $y^{(L)n}$ at global node L. Furthermore, $P_{M,a} \equiv \partial P_m / \partial \xi^a$. Since tensorial character is fully respected, one can easily choose appropriate coordinate system at each global node, useful for prescribing of fluxes and/or temperatures, or results interpretation.

It should be noted that matrix form () represents a new original form of resulting system of linear equations in the steady state heat transfer analysis. Especially, because contribution from convective heat transfer represented by matrix entry D is naturally assembled to specified position in () connected to temperature degrees of freedom.

SOLUTION OF THE RESULTING SYSTEM OF LINEAR EQUATIONS

In accordance to the primal approaches, where temperature (or displacement) is only one unknown variable in the resulting system of linear (finite element) equations, the present mixed finite element approach produces additional number of unknowns connected to the heat flux (or stress) variable. Present system matrix is large, quadratic, symmetric, sparse and indefinite. Therefore, the modified direct MA47 solution procedure, particular version of the Duff-Reid algorithm [3], which solves present type of system equations using multifrontal Gaussian elimination and taking advantage of zero entries on the diagonal, is presently used.

NUMERICAL EXAMPLES

NEARLY INCOMPRESSIBLE BLOCK UNDER COMPRESSION

In the present example [1, 4], we will examine if the finite element HC8/27 is sensitive to Poisson’s effect (volumetric locking) in the limit of incompressibility allied to mesh distortion aligned with non-smooth boundary conditions per stresses [2]. The model problem is brick of side $a = 1mm$ under compression $p_0 = 0.25 N/mm^2$ on the middle part of its two opposite surfaces. Therefore, prescribed pressure field is not smooth over the surface on which it is applied. One octant of the system is discretized due to the symmetry. Geometry, loading and boundary conditions of the system are described in Fig. 1, in which short line represents suppressed displacement along that direction. The nodes on the top of the structure are constrained in x and y direction. The material parameters are $\mu = 80.194\ N/mm^2$ and $\lambda = 400889.806\ N/mm^2$. Consequently, modulus of elasticity is $E = 240.56595979\ N/mm^2$ and Poisson’s ratio is $\nu = .499899987$. Four finite element HC8/27 meshes with increasing refinement $N \times N \times N$, $N = 2,4,6,8$ are analyzed. The finite element HC8/27 has 8 nodes per displacement and 27 per stresses, that is, tri-linear shape approximation functions per displacement and tri-quadratic shape approximation functions per stresses.

It should be added, that present elasticity problems are usually solved [2] by the system of truss finite element or by brick elements where displacement is only one fundamental variable, while stresses are calculated a posteriori which entails a loss of accuracy. On the other hand, the stress is presently also a fundamental variable. Therefore, we have a possibility to have full insight into the real material by merging the present procedure with some numerical method in nano-mechanics.

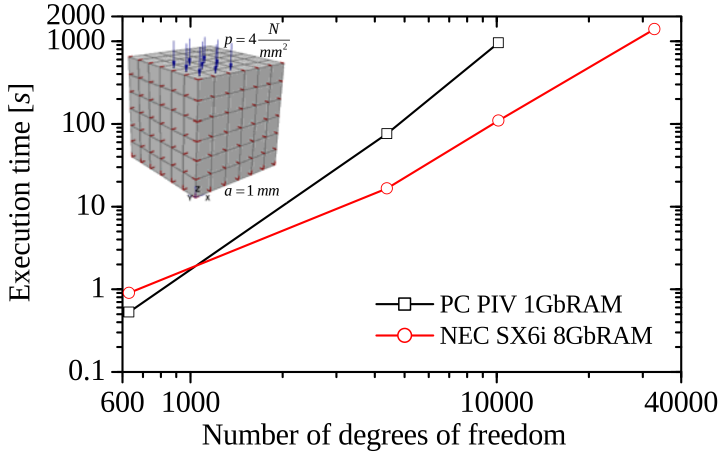

Fig.1. Execution time per mesh refinement

From the results shown in Fig. 1, we can see that using NEC SX6i supercomputer with 8Gb RAM we obtain the results up to one order of magnitude faster than on standard PC, and it also enables us to go finer into the discretisation toward the equation system with 40000 degrees of freedom upon fully tri-quadratic approximation of stress field. It should be noted, that much more degrees of freedom (up to 200000) could be solved with the present computer equipment in a reasonable time (up to 1000 seconds per one direct solution of system of linear equations), but with the low-order finite element HC8/9. Nevertheless, because of the high non-smoothness in data of the present model problem, the stable [1, 5] finite element HC8/27 is chosen because of the accuracy and stability of the solution, which is demonstrated also in Fig.2, where it can be seen that no spurious displacement occurs.

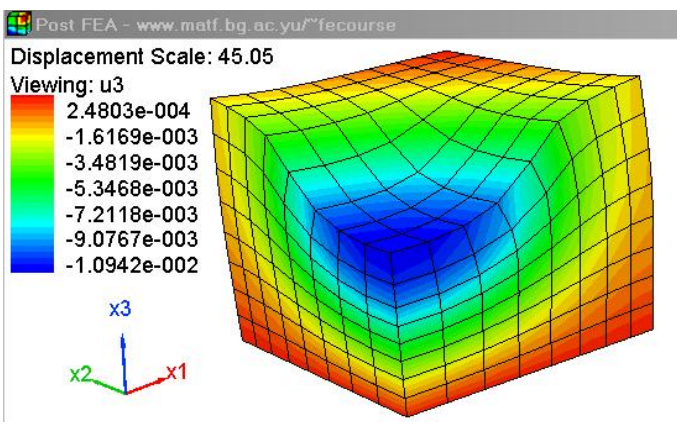

Fig.2. deformed configuration and displacement vector component $u_3 \equiv u_z$

STEADY STATE HEAT TRANSFER THROUGH A COOLING FIN

In the following example, steady state heat transfer through a fin [2, 6], shown in Fig.3, is analyzed. Thermal conductivity of the material is $k = 25\, W/mK$ .One end of the fin is insulated, while the other end is held at a constant temperature of $373\, K$. The fin is surrounded by fluid at constant temperature of $T_a = 273\, K$, and convection coefficient is $h_c = 1\, W/m^2K$. The cross section of the fin has two axes of symmetry, and consequently only one quarter of the model is analyzed. One could go even further in this reduction of number of elements and analyze only 1/8 of the fin, because the method allows different coordinate systems per heat flux at every node. This means that essential and symmetry boundary conditions can be easy imposed on various surfaces.

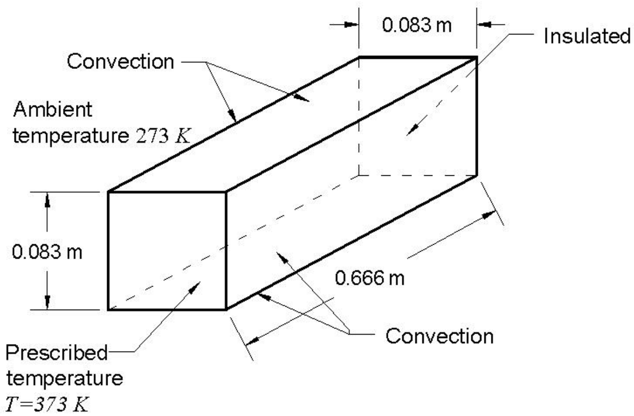

Fig.3. Fin – geometry and boundary conditions of the problem

The execution times in accordance to four consequent finite element discretizations are shown in Fig.4. We have examined finite element meshes with 2, 16, 128 and 1024 finite elements, respectively. The finite element HC8/27 [5] is chosen, which has 8 nodes per temperature, and 27 nodes per heat flux.

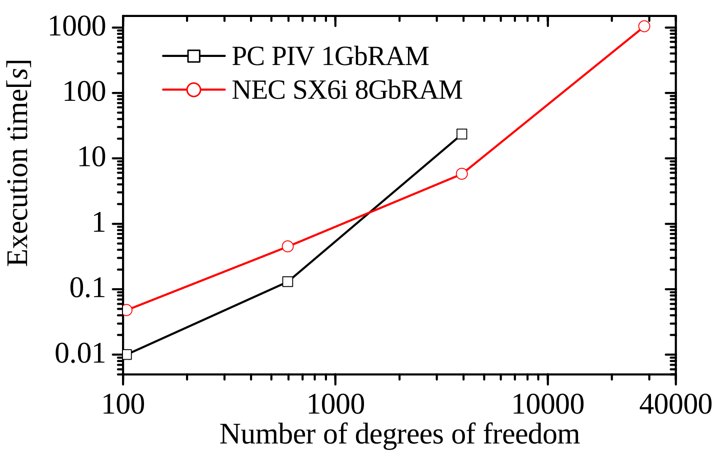

Fig.4. Execution time per mesh refinement

CONCLUSION

From the results shown in Figures 1 and 4, it is clear that by the use of the NEC SX6i computer environment with 8Gb RAM we obtain the results up to ten times faster than by the use of PC Pentium IV 1GbRAM, per each of finer finite element mesh. It can be seen that NEC SX6i performs faster for very large problems, which is caused by its vectorization capabilities, also. Additional memory space available on NEC SX6i enables us to go even further in the mesh refinement process which is the key procedure in stability, that is, convergence investigation of any new finite element approach or finite element analysis.

ACKNOWLEDGMENTS

This investigation is carried under the Grant IO1865 from Ministry of Science and Environmental Protection of Republic of Serbia. The support is gratefully acknowledged. Authors would also like to express deepest gratitude for being granted to work on NEC supercomputer equipment and push the limits of theirs scientific investigations a little further.

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