A NEW MIXED HEXAHEDRAL FINITE ELEMENT IN HEAT TRANSFER ANALYSIS

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Abstract.

A new original primal-mixed finite element approach and related hexahedral finite element $\mathrm{HC}{:}T/q$ *for the analysis of behavior of solid bodies under thermal loading is presented here. The essential contributions of the present approach are the treatment of temperatures and heat fluxes as fundamental variables, further the solution of temperatures and heat fluxes from the same system of linear equations, and initial/prescribed temperature and heat flux capability. In order to minimize accuracy error and enable introductions of flux constraints, the tensorial character of the present finite element equations is fully respected. The proposed finite element is subjected to the standard benchmark test in order to test convergence of the results, which enlighten the effectiveness and reliability of the approach proposed.*

Keys-words*: steady sate heat, finite elements, mixed formulation*

Introduction

In the present paper a new original finite element approach for the solution of the steady state heat transfer in the solid body is presented. The main motive for the present investigation is found in lack of hexahedral finite element which is reliable [1] and robust in accordance to change of its aspect ratio, and simultaneously, finite element procedure which treats both variable of interest, temperature and heat flux, as fundamental ones [2]. Further, the motive is also found in the known problem of connecting the finite elements of different dimensionality, i.e. when a model problem has geometrical transitions from solid to thick or thin.

Further, the main objective of the present investigation is to show that a new reliable [1] mixed hexahedral (brick) finite element $\mathrm{HC}{:}T/q$ [2], may be used in the analysis engineering constructions of arbitrary shape, without need for *a posteriori* calculation of heat fluxes. Thus, fort the difference to the primal approaches, present finite element approach has two fundamental variables: temperature and heat fluxes.

Consequently, the main goal of the present paper is to validate the use of finite element $\mathrm{HC}{:}T/q$ in the steady state heat analysis of isotropic, orthotropic or multi-materials solid bodies under the different thermal or mechanical loading scenarios.

Nevertheless, it is planned to implement the present approach in the existing in-house primal-mixed elasticity code for the more accurate determination of thermal stresses, where no consistency problems will occur in calculation of thermal and mechanical deformations [3, 4].

1 Weak form of the steady state heat field equations

Consider a body which occupies the closed and bounded domain $\bar{\Omega}$ of the Euclidian space $E^n\ (n = 1,2,3)$. The inner part of $\bar{\Omega}$ is denoted by Ω and its boundary by $\partial\Omega$, $\Omega \cup \partial\Omega = \bar{\Omega}$. The boundary is subdivided into four parts $\partial\Omega_T$, $\partial\Omega_q$, $\partial\Omega_c$, $\partial\Omega_r$ such that $\partial\Omega_T \cup \partial\Omega_q \cup \partial\Omega_c \cup \partial\Omega_r = \partial\Omega$. The state of the body is described by the temperature T and by the heat flux vector $\boldsymbol{q}$. Contact thermal flux through an interface is $\boldsymbol{q}\cdot\boldsymbol{n}$, where $\boldsymbol{n}$ is the unit outward normal vector on the interface.

Let us consider a complete system of the field equations in the steady-state heat transfer where,

$$\mathrm{div}\,\boldsymbol{q} + f = 0 \ \text{ in } \ \Omega\,, \tag{1}$$

$$\boldsymbol{q} = -\boldsymbol{k}\nabla T \ \text{ in } \ \Omega\,, \tag{2}$$

are respectively the equations of thermal balance that states that the divergence of the heat flux is equal to the internal heat source f, Fourier law of heat conduction which assumes that the heat flux is linearly related to the negative gradient of the temperature, where $\boldsymbol{k}$ is second order tensor of thermal conductivity.

These two equations are subjected to the next boundary conditions:

$$T = \bar{T} \ \text{ on } \ \partial\Omega_T\,, \tag{3}$$

$$\boldsymbol{q}\cdot\boldsymbol{n} = q_h = h \ \text{ on } \ \partial\Omega_q\,, \tag{4}$$

$$\boldsymbol{q}\cdot\boldsymbol{n} = q_c = h_c(T - T_0) \ \text{ on } \ \partial\Omega_c\,, \tag{5}$$

$$\boldsymbol{q}\cdot\boldsymbol{n} = q_r = h_r(T^4 - T_0^4) \ \text{ on } \ \partial\Omega_r\,, \tag{6}$$

Firstly, boundary condition per temperature (3) which is prescribed to be equal to $\bar{T}$ over a portion $\partial\Omega_T$ of the boundary $\partial\Omega$ and to heat flux. Secondly, boundary conditions due to the prescribed heat flux on boundary (4), next due to the convection (5), where h_c is the convective coefficient and T_0 is the temperature of the surrounding medium, and finally, due to the radiation (6),which will not be considered because they are source of nonlinearity.

Let us suppose that all boundary conditions (3), (4), (5) are essential, and hence exactly satisfied by the trial functions of a problem. Then we need to consider only the weak forms of the equations (1) and (2). By the use of the Galerkin procedure, one can seek the weak solution of (1)

$$\int_\Omega (div\boldsymbol{q} + f)\Theta d\Omega = 0 \tag{7}$$

where Θ is taken from the Hilbert space L_2 of all real measurable square integral scalar functions. Further, we will consider the weak form of invertible constitutive equations (1.2), where $\boldsymbol{Q}$ is the test function taken from the space of all measurable square integrable vector fields:

$$\int_\Omega (\mathbf{k}^{-1}\boldsymbol{q} + \nabla T)\boldsymbol{Q} d\Omega = 0 \tag{8}$$

By the simple summation of (7) and (8) we obtain the expression which represents asymmetric weak formulation of a mixed problem. However, it is presently a common sense that asymmetric formulations are impractical from the computational point of view. Integrating by parts and applying divergence theorem on the first term on the left side yields symmetric weak form of a mixed problem:

Find $\{T,\boldsymbol{q}\} \in H^1(\Omega)\times L_2(\Omega)$ such that $T\big|_{\partial\Omega_T} = \bar{T}$ and

$$\int_\Omega \boldsymbol{q}\mathbf{k}^{-1}\boldsymbol{Q} d\Omega + \int_\Omega \nabla T\cdot\boldsymbol{Q} d\Omega = \int_\Omega \boldsymbol{q}\nabla\Theta d\Omega - \int_\Omega \Theta f d\Omega - \int_{\partial\Omega_q} \Theta h d\partial\Omega - \int_{\partial\Omega_c} q_c\Theta d\partial\Omega \tag{10}$$

for all $\{\theta,\boldsymbol{Q}\} \in H^1(\Omega)\times L_2(\Omega)$ such that $\theta\big|_{\partial\Omega_T} = 0$,

where H^1 is the space of all scalar fields which are square integrable and have square integrable gradient.

2 Finite element approximations of the field equations

Let C_h be the partitioning of the domain Ω into elements Ω_i and let us define the finite element subspaces for the temperature scalar T, the heat flux vector $\boldsymbol{q}$ and the appropriate weight functions, respectively as:

$$\begin{aligned}
T_h &= \left\{ T \in H^1(\Omega) : T|_{\partial\Omega_T} = \bar{T},\ \ T = T^L P_L(\Omega_i), \forall \Omega_i \in C_h \right\} \\
\Theta_h &= \left\{ \Theta \in H^1(\Omega) : \Theta|_{\partial\Omega_T} = 0,\ \ \Theta = \Theta^M P_M(\Omega_i), \forall \Omega_i \in C_h \right\} \\
Q_h &= \left\{ \boldsymbol{q} \in H^1(\Omega) : \boldsymbol{q}\cdot\boldsymbol{n}|_{\partial\Omega_q} = h,\ \ \boldsymbol{q}\cdot\boldsymbol{n}|_{\partial\Omega_c} = h_c(T - T_0), \boldsymbol{q} = \boldsymbol{q}^L V_L(\Omega_i), \forall \Omega_i \in C_h \right\} \\
\vartheta_h &= \left\{ \boldsymbol{Q} \in H^1(\Omega) : \boldsymbol{Q}\cdot\boldsymbol{n}|_{\partial\Omega_q \cup \partial\Omega_c} = 0,\ \ \boldsymbol{Q} = \boldsymbol{Q}^M V_M(\Omega_i), \forall \Omega_i \in C_h \right\}
\end{aligned} \tag{11}$$

In these expressions T^L and $\boldsymbol{q}^L$ are the nodal values of the temperature scalar T and flux vector $\boldsymbol{q}$, respectively. Accordingly, P_L and V_L are corresponding values of the interpolation functions, connecting temperatures and fluxes at an arbitrary point in Ω_i (the body of an element), and the nodal values of these quantities. The complete analogy holds for the temperature and flux weight functions Θ and $\boldsymbol{Q}$.

It should be noted that when present finite element approach is applied on model problems with abrupt material changes where local heat flux discontinuity is possible to exist, the present rule that local stress approximation function are from continuous function space $\left(H^1\right)^{n\times n}$ is to hard. It is left for the future investigation to relax stress continuity on the interface surface(s), where fluxes will be chosen from space $L^2(\Omega)^{n\times n}_{\mathrm{sym}}$, as in original primal-mixed formulation.

3 Numerical implementation

By analogy with finite element approach in elasticity [5], after discretization of the starting problem by finite element method, present scheme can be written as the system of linear equations of order $n = n_q + n_T$, where n_T is the number of temperature degrees of freedom, while n_q is the number of flux degrees of freedom:

$$\begin{bmatrix} A_{vv} & B_{vv}{}^T \\ B_{vv} & -D_{vv} \end{bmatrix} \begin{bmatrix} q_v \\ T_v \end{bmatrix} = \begin{bmatrix} -A_{vp} & -B_{vp}{}^T \\ -B_{vp} & D_{vp} \end{bmatrix} \begin{bmatrix} q_p \\ T_p \end{bmatrix} + \begin{bmatrix} 0 \\ F_p + H_p - K_p \end{bmatrix} \tag{12}$$

In this expression, unknown (variable) and known (initial, prescribed) values of the fluxes and temperatures, denoted by the indices v and p respectively, are decomposed.

The nodal fluxes q^{pL} and temperatures T^L components are consecutively ordered in the column matrices $\boldsymbol{q}$ and $\boldsymbol{T}$ respectively. The homogeneous and nonhomogenous essential boundary conditions per temperatures T_p and fluxes q_p are introduced as contribution to the right-hand side of the expression (12).

The members of the matrices **A**, **B** and **D**, of the column matrices **F, H**, and **K** (discretized body and surface forces) in (12), are respectively:

$$\begin{aligned}
A_{LpMr} &= \sum_e \int_{\Omega_e} g^a_{(L)p} V_L r_{ab} g^b_{(M)r} V_M \, d\Omega_e; \quad B_{LpM} = \sum_e \int_{\Omega_e} g^a_{(L)p} V_L P_{M,a} \, d\Omega_e \\
D_{LM} &= \sum_e \int_{\partial\Omega_{ce}} h_c P_L P_M \partial\Omega_{ce}; \quad F_M = \sum_e \int_{\Omega_e} P_M f \, d\Omega_e \\
H_M &= \sum_e \int_{\partial\Omega_{he}} P_M h d\partial\Omega_{he}; \quad K_M = \sum_e \int_{\partial\Omega_{ce}} P_M h_c T_0 d\partial\Omega_{ce}
\end{aligned} \tag{2}$$

In the above expressions, r_{ab} are the components of the inverse of the second order tensor of the thermal conductivity given by the $\boldsymbol{k} = k^{ab} \boldsymbol{g}_a \otimes \boldsymbol{g}_b$, while the Euclidian shifting operator $g^a_{(L)p}$ is given by $g^a_{(L)p} = \delta_{ij} g^{ac} \frac{\partial z^i}{\partial \xi^c} \frac{\partial z^j}{\partial y^{(L)p}}$, where, $z^i (i,j,k,l = 1,2,3)$ is the global Cartesian coordinate system of reference. Further, $y^{(L)r} (r,s,t = 1,2,3)$ is coordinate system at each global node L, per heat flux. Further, the local natural (convective) coordinate systems per finite elements are denoted by $\xi^a (a,b,c,d = 1,2,3)$, while, g^{ab} and $g^{(L)mn}$ are the components of the contravariant fundamental metric tensors, the first one with respect to natural coordinate system of a finite element ξ^a, and the second to $y^{(L)n}$ at global node L. Furthermore, $P^M_{,a} \equiv \frac{\partial P^M}{\partial \xi^a}$. For the reason that tensorial character is fully respected, one can easily choose appropriate coordinate system at each global node for the introductions of known fluxes and/or temperatures, or interpretation of the results.

4 Finite element HC8/27

The finite element HC8/27 is shown in Figure 1. Its acronym is taken from [14], where the first letter H stands for hexahedral element geometrical shape, while the letter C indicates the use of continuous approximation functions. The temperature nodes are denoted by spheres, while heat flux nodes are denoted by tetrahedrons, as shown in Figure 1.

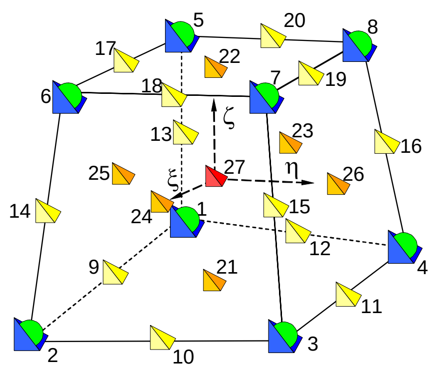

Fig. 1. Finite element HC8/27

The temperature and heat flux filed is approximated by tre-linear shape functions, connected to eight corner nodes. In addition, stabilization of finite element is achieved by full or partial hierarchic interpolation of heat flux one order higher than temperature connected to the additional up to nineteen nodes. Thus, twenty-seven stress nodes are available to accommodate full triquadratic expansion in natural coordinates ξ, η and ζ. Consequently, the next multifield combinations of the temperature and flux nodes are available to user: HC8/9, HC20/21, HC8/27 and HC20/27. Moreover, per each corner node there is maximum four degrees of freedom $n = 4$ (one temperature degree of freedom $n_T = 1$ and three degrees of freedom pre heat flux $n_q = 3$), while in hierarchical nodes (face centre, mid-side and bubble) there are only degrees of freedom connected to heat flux.

5 Solution of the resulting system of linear equations

In the present paper, the method based on the multifrontal approach, one of the main categories of direct methods for the solution of the resulting system of linear equations, is used. The core of that method is taken from the code MA47 [6], representing a version of sparse Gaussian elimination which is implemented using a multifrontal method.

6 Numerical experiment

6.1 Steady State Heat Transfer in a Solid Steel Billet

In the present example [6] the steady state heat transfer in a solid steel billet, shown in Fig. 2, is analyzed. The target value is temperature at point A: $T^{A} = 32.8\,^{0}C$. The material properties are $50\,W/m^{\circ}C$ and $100\,W/m^{\circ}C$ for thermal conductivity and heat transfer coefficient, respectively. Ambient bulk fluid temperature is, like in the first model problem, of $0\,^{\circ}C$.The model is discretisized by increasing sequence of the refinement factor, $N \times 3N \times L$, where $N = 4,6,8,10,12,14$, and $L = 3,4,5,6,7,8$, in order to check the convergence of the finite element solutions.

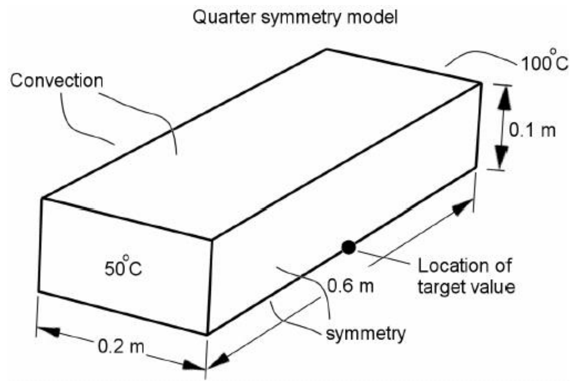

Figure 2: Solid steel billet

From the temperature results at the target point shown in Figure 3, we may see that both finite element approaches, present and primal, converge uniformly to the same value which is a little bit lower than target value.

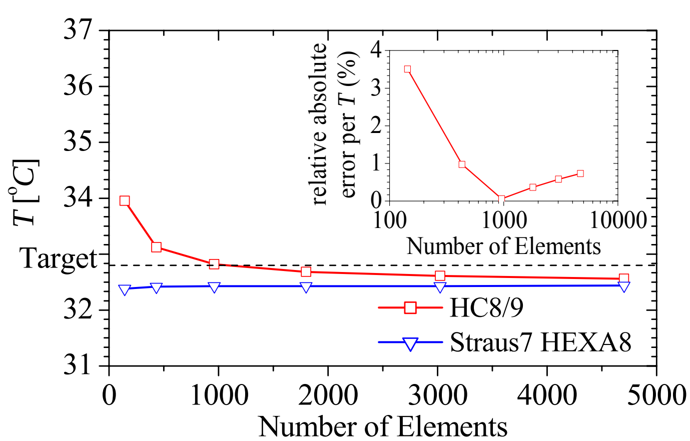

Figure 3: Solid billet: The convergence of temperature at target point

The main difference is in calculation of heat flux field, see Figure 4. It is calculated *a posteriori* in the examined primal approach (Straus7) resulting with abnormal discontinuity along element interfaces, which raise the need for the use of some recovery or smoothing technique of the heat flux (dual) variable [8]. On the other hand, presently heat flux filed is calculated as the fundamental variable, and it is continuous as it is expected to be.

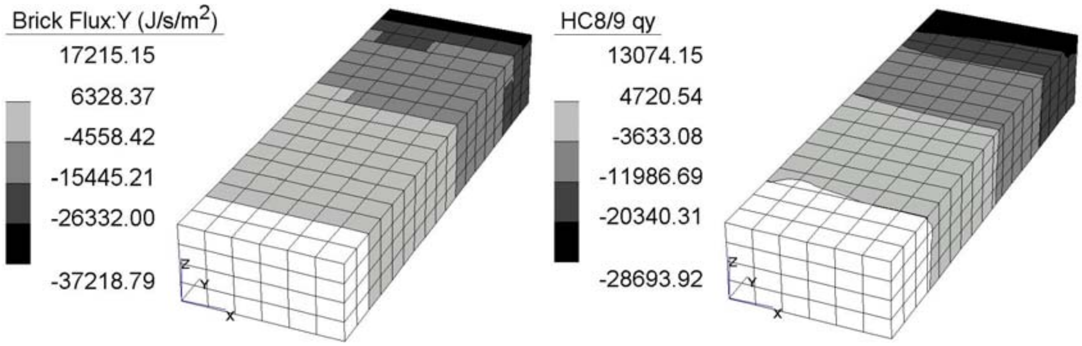

Figure 3: The heat flux q^y calculated by the finite elements HEXA20 and HC8/9

Conclusion

From the standard benchmark example in steady sate heat analysis of solid bodies solved by the present hexahedral finite element HC8/9, we may preliminary conclude that it has good convergence. Nevertheless, the detailed investigation of all aspects of convergence, as consistency and stability requirements are [1], is left for further investigation. Moreover, we may emphasize that one of the main potential of the present finite element is that it use overcomes known transition problem of connecting finite elements of different type and dimensions. Consequently, we may end with the conclusion that present finite element approach give us greater design freedom than standard primal approaches that use different kind of finite elements: solid, plate/shell, beam... In addition, the present finite element approach will be used in connection with the existing in-house software [5], based on the original reliable mixed u/σ finite element approach in elastic analysis, for determination of the thermal stresses, which is left for further report, also.

Acknowledgements

The author is grateful to the Ministry of Science, Technologies and Development of Republic of Serbia for providing financial support for this research, under the projects No.IO1865 and EE197b.

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