Chapter 1 Primal finite element approach

To examine the behavior of the solid body under mechanical load $\boldsymbol{F}$ (see Figure 1.1) we may start with the fundamental problem of elastostatics theory in classical notation given by equilibrium equation:

$$\operatorname{div}\boldsymbol{t}+\rho\boldsymbol{b}=\boldsymbol{0} \quad \text{in } \Omega\,, \tag{1.1}$$

and displacement boundary conditions (b.c), which can be zero (homogenous) and nonzero (nonhomogenous):

$$\boldsymbol{u}=\overline{\boldsymbol{u}} \text{ on } \partial\Omega_u\,. \tag{1.2}$$

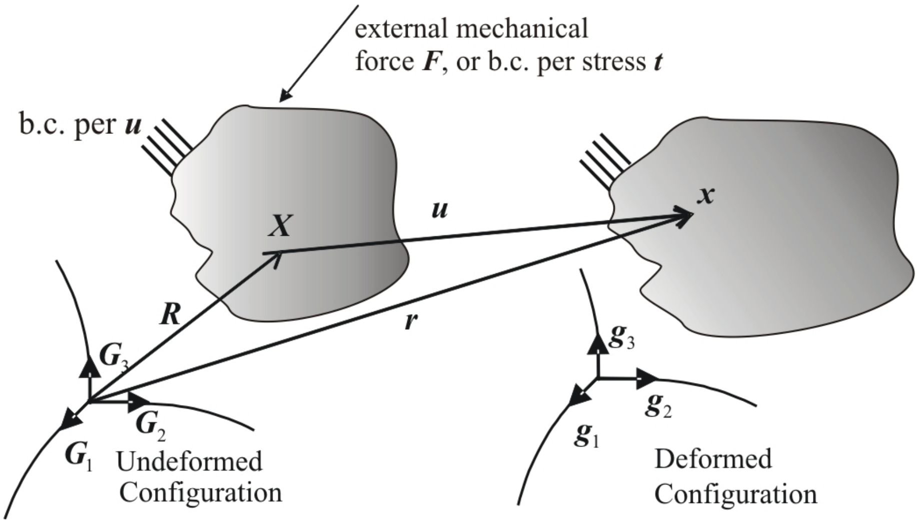

Figure 1.1. Solid body under deformation due to the arbitrary mechanical load

Additionally, boundary conditions per stress, i.e. boundary tractions $\boldsymbol{p}$, over the part of the boundary $\partial\Omega_t=\partial\Omega\setminus\partial\Omega_u$ may be suppressed (zero (homogenous)) or prescribed (nonzero (nonhomogenous)):

$$\boldsymbol{t}\cdot\boldsymbol{n}=\boldsymbol{p} \text{ on } \partial\Omega_t \tag{1.3}$$

If it is physically justified that we assume that considered body during deformation process behaves geometrically linear (i.e. displacements are *small*), continuity equation which relates displacements $\boldsymbol{u}$ with second order strain tensor $\boldsymbol{e}$ over the domain of the body Ω is given by next expression:

$$\boldsymbol{e}=\frac{1}{2}\left(\operatorname{grad}\boldsymbol{u}+\operatorname{grad}\boldsymbol{u}^{\mathrm{T}}\right) \quad \text{in } \Omega\,. \tag{1.4}$$

In order to have complete system of equation we need the constitutive equation which gives us functional dependence between stress and strain due to the mechanical load:

$$\boldsymbol{t} = \mathbf{C}\,\boldsymbol{e}\,, \tag{1.5}$$

Let's emphasized that in order to obtain the unique solution of the problem (1.1) we must have prescribed boundary condition per displacement or stress over the part of the boundary $\partial\Omega$.

In the above equations, $\boldsymbol{t}$ is the Cauchy second order stress tensor, where holds: $\boldsymbol{t} = t^{ab}\boldsymbol{g}_a \otimes \boldsymbol{g}_b$. Next, ρ is the material density, and $\boldsymbol{b}$ are the body forces. The $\Omega \subset R^d, d = 1,2,3$ is open domain of the considered body, where d being geometrical dimension of the problem. $\partial\Omega_u$ and $\partial\Omega_t$ are parts of the boundary where homogenous or nonhomogenous boundary conditions per displacements and stresses are prescribed, respectively. Further, $\mathbf{C}$ is the elasticity fourth order tensor, is given by $\mathbf{C} = C^{abcd}\boldsymbol{g}_a \otimes \boldsymbol{g}_b \otimes \boldsymbol{g}_c \otimes \boldsymbol{g}_d$ assumed to be symmetric and positive definite. Further, $\boldsymbol{G}_i$ is the base vector of coordinate system z^i in initial configuration, while $\boldsymbol{g}_a$ is the base vector of spatial coordinate system ξ^a in deformed configuration.

For applying a finite element method one has to reformulate problem (1.1) in a variational setting. Let's denote by $\boldsymbol{u}$ the trial (unknown) function. Next, let's $\boldsymbol{v}$ be a test function of a displacement solution variable with property that it should vanish on the boundary. The primal variational formulation of problem (1.1) is obtained by writing the equilibrium law in a weak form and integrating it by parts:

$$\int_{\Omega} \nabla\boldsymbol{u}\,\mathbf{C}\,\nabla\boldsymbol{v}\; d\Omega = \int_{\Omega} \rho\boldsymbol{b}\cdot\boldsymbol{v}\; d\Omega + \int_{\partial\Omega} \boldsymbol{p}\cdot\boldsymbol{v}\; d\partial\Omega \tag{1.6}$$

Now we will transform our starting problem given in an infinite dimensional space to one given in finite dimensional space. The domain of the body Ω will be discretized by the finite element subdomains of beam like, shell like, or brick or tetrahedral like shapes. That is, the domain Ω will be discretized by simple geometrical objects, subdomains Ω_e, in order to ease the approximation of the domain and boundary.

Over each of the subdomain Ω_e we will define linear, quadratic or higher order polynomial shape (base, approximation) functions $P_L\left(\xi^\alpha\right)$ with local support. Namely, if P_L and V_L are two basis functions associated with the nodes N_1 and N_2, then the supports of the functions P_L and V_L overlap only if N_1 and N_2 belong to the same element.

We may have different types and orders of base functions over the finite element mesh providing that *continuity condition* [2] holds. The finite element has finite number of local nodes $L = 1, NLN$, where NLN is the number of local nodes in finite element (see Figure 1.3).

Now we may define finite element approximation $\boldsymbol{u}_h$ of the solution variable, in present case displacement vector $\boldsymbol{u}$, over the finite element model with domain $\Omega_h = \sum_e \Omega_e \approx \Omega$:

$$\boldsymbol{u}_h = \sum_e \boldsymbol{u}^L P_L, \quad L = 1, NLN \tag{1.7}$$

By introducing (1.7) in (1.6) we obtain the well known matrix equation of primal finite element method, known widely as displacement finite element method, given by:

$$[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{F}\}. \tag{1.8}$$

Above, $[\mathbf{K}]$ is a global stiffness matrix, $\{\mathbf{u}\}$ the solution vector and $\{\mathbf{F}\}$ is a global node load vector. Matrix entries in (1.8) are calculated by:

$$K^{\Lambda s \Gamma t} = \int_{\Omega_e} \overset{(e)}{\Omega}{}^{\Lambda}_{L} \, P^L_a \, g^{(L)s}_b C^{abcd} g^{(K)t}_d \, P^K_c \, \overset{(e)}{\Omega}{}^{\Gamma}_{K} \, d\Omega \tag{1.9}$$

$$F^{\Lambda q} = \sum_e \int_{\Omega_e} g^{(\Lambda)q}_a \, \overset{(e)}{\Omega}{}^{\Lambda}_{M} \, P^M b^a d\Omega + \sum_e \int_{\partial\Omega_e} g^{(\Lambda)q}_a \, \overset{(e)}{\Omega}{}^{\Lambda}_{M} \, P^M p^a d\partial\Omega \tag{1.10}$$

In the above expressions and further in the text Einstein convention of summation on repeated indices is used. Euclidian shifters between convective, natural, coordinate system of finite element ξ^α, $\alpha = 1,2,3$ and coordinate system at node Λ, are given by $g^a_{(\Lambda)u}$ or $g^{(\Lambda)q}_a$. It holds: $P^K_a = \partial P^K / \partial \xi^a$. Further, b^a and p^a are the body forces and boundary tractions in natural coordinates of an element, respectively.

Tensorial invariance of the above expressions will enable us to choose different coordinate systems at the global nodes per displacements and stresses and to select proper positions of nodal coordinate systems for introduction of stress and displacement constraints.

Because of the property that base functions have local support, the system matrix $[\mathbf{K}]$ is sparsely populated by nonzero entries, that is, it is sparse, with much more zero than nonzero entries. In the sparsisity of matrix equations lies main success of the finite element approach. Namely, sparsisity has good influence on the decreasing of the solution time. The lesser number of nonzero entries in each row of the resulting system matrix, the solution time in general will be smaller. Otherwise, the greater number of local nodes in each finite element, the number of nonzero entries in corresponding rows in system matrix will be greater. From that reason, there was a great interest for low–order elements in past decades. By (1.8) we now can solve the equations of equilibrium for the unknown nodal displacements.

Other field variables of interest, such as, strains, stresses, stress resultants, strain energy densities, etc., remain to be calculated by some of the *a posteriori* routines, so-called *recovery* routines. [15, 16], which entails a lost of accuracy [17].

In addition, although it is sometimes very important in engineering calculations, the *a priori* known residual stress field or initial strain field can not be introduced directly, i.e. without differentiation, in (1.8).

If structural behavior of some engineering construction has to be analyzed by the finite element method [13, 14], the next steps should be performed consecutively (see Figure 1.2):

- *Discretize* domain of the body, external loads, and boundary conditions,
- *Approximate* material properties, that is, real material with imperfections should be represented as an isotropic, orthotropic, or composite or functionally graded material, with scale resolution up to at most micron size only. It means that we will observe the body from the continuum mechanics point of view.
- *Assume* best fit constitutive equations for materials of the body.

Under structural behavior in the present context we assume the determination of the displacement and strain or stress field of the thermo-mechanically loaded solid body under the consideration (see Figure 1.2).

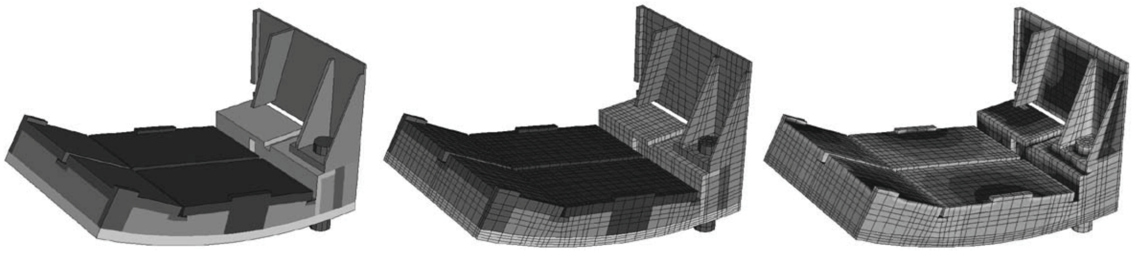

Figure 1.2. The section of the rotary lime kiln: model problem, FE mesh and stress field.

1.1 Finite element and related shape functions

When a mesh of simplicial elements (triangles or tetrahedra) is used to form a piecewise linear approximation of a function, the accuracy of the approximation depends on the sizes and shapes of the elements. In finite element methods, the conditioning of the stiffness matrices also depends on the sizes and shapes of the elements. There are strong mathematical connections between mesh geometry, interpolation errors, discretization errors, and stiffness matrix conditioning.

The generalization of quadrilateral in three-dimensional space is a hexahedron, more popularly known in the finite element literature as brick. A brick is topologically equivalent to a cube. It has eight corners, twelve edges or sides, and six faces. Finite elements with this geometry are extensively used in modeling three-dimensional solids. Bricks also have been the motivating factor for the development of “Ahmad-Pawsey” shell elements through the use of the “degenerated solid” concept.

The construction of brick shape functions and the computation of the stiffness matrix was enormously facilitated by three advances in finite element technology: natural coordinates, isoparametric description and numerical integration, which revolutionized the finite element field in the mid-1960’s.

Regardless of the shape or topology of the finite elements used in the approximation process there are two possibilities to refine the approximation, so-called p and h refinement. The former refers to the refinement of the order of the finite element shape functions used, while the later refers to the increasing of the number of the finite elements while simultaneously decreasing its size. It is possible to have combination of these two approaches. The example of that is the approximation by the *hierarchic* shape function [8], which is used in the present text.

1.2 Coordinate independence of the finite element equations

Some physical low is a law of the nature only if it is independent on the coordinate system in which it is interpreted [18], that is, physical laws are given by the tensorial equations. To avoid introduction of the *geometric invariance* error [19] in the finite element approximation process of the physical laws of mechanical and thermal fields and corresponding constitutive and continuity equations used in the present text, and as well to enable introduction of primal and dual (displacement and stress, temperature and heat flux) boundary conditions in arbitrary coordinates, the invariant derivation of the present finite element equations in curvilinear coordinates is presently strongly respected [20].

References

[1] Šilhavý M (1997) The Mechanics and Thermodynamics of Continuous Media, Springer

[2] Jarić J (1988) Continuum Mechanics (In Serbian), Građevinska knjiga, Beograd

[3] Arnold DN (1990) Mixed finite element methods for elliptic problems, Computer Methods in Applied Mechanics and Engineering, 82:281-300

[4] Vujanović B, Atanacković T (2004) An Introduction to Modern Variational Techniques in Mechanics and Engineering, Springer

[5] Bathe KJ (1982) Finite element procedure, Prentice Hall.

[6] Kojić M (1997) Computational procedures in inelastic analysis of solids and structures, Center for Scientific Research of SASU and University of Kragujevac

[7] Jovanović B (1993) The finite difference method for boundary-value problems with weak solutions, Mathematical Institute SASA (1993)

[8] Zienkiewicz OC, Taylor RL (2000) The finite element method – Volume I – The basis, Butterworth-Heinemann

[9] Mijuca D (2004) On hexahedral finite element HC8/27 in elasticity Computational Mechanics, Springe-Verlag, Vol 33, 6:466-480

[10] Mijuca D, Mastilović S (2005) A Novel One-To-One Multiscale Approach to Computational Mechanics of Materials, , 1st International Workshop on Nanoscience & Nanotechnology IWON 2005 and 4th COSENT Annual Meeting Belgrade, November 15-18, pp. 180-86

[11] Mijuca D, Žiberna A, Međo B (2006) A Novel Primal-Mixed Finite Element Approach for Heat Transfer in Solids, Computational Mechanics, on-line first DOI 10.1007/s00466-006-0034-0

[12] Ghoniemy NM, Busso EP, Kioussis N, Huang H (2003) Multiscale modelling of nanomechanics and, micromechanics: an overview. Philosophical Magazine, 1 Vol. 83, Nos. 31–34:3475–3528

[13] G+D Computing Straus7, Finite element analysis system software package – Verification Manual, www.strand.aust.com, Australia

[14] Mijuca D (2003) Three-dimensional finite element analyses of lime kiln upgrade recuperation by displacement and mixed finite element methods – a comparison. Professional report 2003/01.

[15] Hinton E, Campbell JS (1974) Local and global smoothing of discontinuous finite element functions using a least squares method. Int.J.Num.Meths. Eng. 8:461–480

[16] Riggs HR, Tessler A (1994) Continuous versus wireframe variational smoothing methods for finite element stress recovery", Advances in Post and Postprocessing for Finite Element Technology, CIVIL-COMP Ltd, Edinburgh, Scotland, pp. 137–144

[17] Mijuca D, Berković M (1997) Stress recovery procedure based on the known displacement. Facta Universitatis, Series Mechanics, Automatic control and Robotics, 7(2):513-523

[18] Andjelić T (1980) Tensor calculus (In Serbian), Naučna knjiga, , Beograd

[19] Bottasso C L, Borri M, Trainelli L (2002) Geometric invariance, Computational Mechanics 29:163-169 DOI 10.1007/s00466-002-0329-8

[20] Drašković Z (2005) On the invariance in mechanics, Matematički institut SANU