Linearized Elasticity

- Cauchy stress is related to infinitesimal strain via a tensor of elastic constants.
- Generalized Hookes Law:
  - anisotropic - 81 coefficients, only 21 unique because of symmetries:

$$T_{ij} = \mathsf{C}_{ijkl} e_{kl} \quad \Leftrightarrow \quad \boldsymbol{T} = \mathsf{C} : \boldsymbol{e}$$

  - Isotropic - can represent the material behavior via two coefficients:

$$T_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} \quad \Leftrightarrow \quad \boldsymbol{T} = \lambda \, \mathrm{tr}(\boldsymbol{e}) \boldsymbol{1} + 2\mu \boldsymbol{e}$$

  - λ and μ are referred to as the Lamé coefficients

Summary - Elasticity Engineering Constants

- It is straightforward to convert between any two of the elasticity constants. Any of the following pairs uniquely defines the material behavior of an isotropic linear elastic material:

$\lambda, \quad \mu \quad$ (Lame′ coefficients)

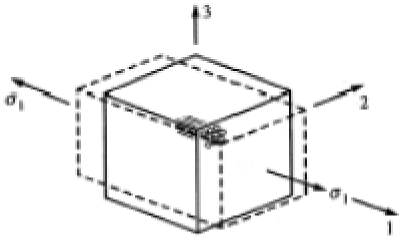
$E, \quad \nu \quad$ (Young's modulus, Poisson's ratio)

$G, \quad K \quad$ (Shear modulus, bulk modulus)

Youngs's Modulus

- The Young's modulus is a material coefficient controlling the linear relationship between an applied uniaxial tensile or compressive stress and the resulting strain in the same direction.
- For a cube of an orthotropic material subjected to a uniaxial stretch along the 1-direction:

$$T_{11} = E_1 e_{11}$$

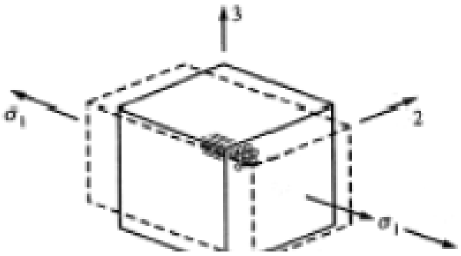


Poisson's Ratio

- Controls the linear relationship between the contraction in a direction transverse to that of an applied uniaxial tensile or compressive stress and the strain along the axis of applied loading.
- For a cube of an orthotropic material subjected to a uniaxial stretch along the 1-direction:

$$\nu_{12} = \frac{-e_{22}}{e_{11}}$$

- Here, e_{22} is the strain in the **2** direction and e_{11} is the strain in the ***1*** direction in a uniaxial stress test along the ***1*** direction.
- So, first subscript refers to direction of applied tensile stress, and second subscript refers to the directs of contraction.

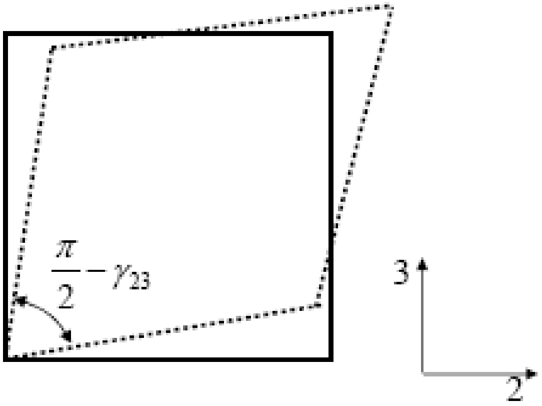


Shear Modulus

- Controls the linear relationship between applied shear stress and resulting shear strain:

- For a cube of an orthotropic material subjected to a shearing strain in the 2-3 plane:

$$T_{23} = G_{23}\gamma_{23}$$



"Tensorial" vs. Engineering Shear Strain

- Infinitesimal strain is defined as:

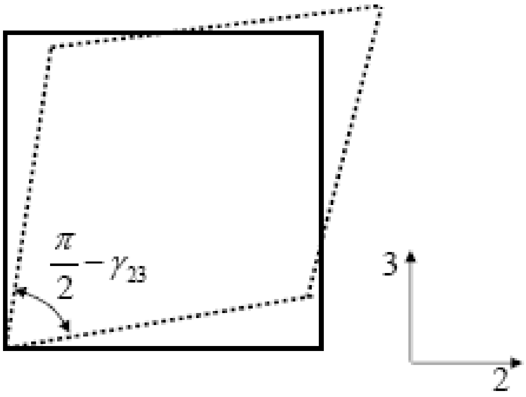
$$e_{IJ} = \frac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I}\right) \Leftrightarrow e = \frac{1}{2}\left(\frac{du}{dX} + \left(\frac{du}{dX}\right)^T\right)$$

- The shear strains are then:

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right), \quad e_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right), \quad e_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right)$$

"Tensorial" vs. Engineering Shear Strain (cont'd)

- Engineering shear strain (γ_{12}, γ_{13}, γ_{23}) is based on the observation that the change in angle under a shearing strain is directly related to 2 times the tensorial shear strain:



$$\gamma_{12} = 2e_{12}$$
$$\gamma_{13} = 2e_{13}$$
$$\gamma_{23} = 2e_{23}$$

Voigt Notation for Linearized Elasticity

- Recognizes that stress and strain tensors each have only six unique components and that both tensors are symmetric. Then the 4th order elasticity tensor can be reduced from 81 coefficients to 36 (21 unique coefficients in general since the elasticity tensor in this notation is symmetric), and written as a 6x6 matrix:

$$[T]=\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{13} \end{bmatrix},\ [e]=\begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix},\ [\mathsf{C}]=\begin{bmatrix} \mathsf{C}_{1111} & \mathsf{C}_{1122} & \mathsf{C}_{1133} & \mathsf{C}_{1112} & \mathsf{C}_{1123} & \mathsf{C}_{1113} \\ & \mathsf{C}_{2222} & \mathsf{C}_{2233} & \mathsf{C}_{2212} & \mathsf{C}_{2223} & \mathsf{C}_{2213} \\ & & \mathsf{C}_{3333} & \mathsf{C}_{3312} & \mathsf{C}_{3323} & \mathsf{C}_{3313} \\ & & & \mathsf{C}_{1212} & \mathsf{C}_{1223} & \mathsf{C}_{1213} \\ & \text{(symmetric)} & & & \mathsf{C}_{2323} & \mathsf{C}_{2313} \\ & & & & & \mathsf{C}_{1313} \end{bmatrix}$$

$$\Rightarrow [T]=[\mathsf{C}]\cdot[e] \text{ or } [e]=[\mathsf{C}]^{-1}\cdot[T]$$

- This is the Voigt notation representation of generalized Hooke's law

Engineering Constants (cont'd)

- Shear modulus, G = half the slope of the shear stress vs. shear strain curve:

$$T_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \Leftrightarrow \boldsymbol{T} = \lambda\,\mathrm{tr}(\boldsymbol{e})\boldsymbol{1} + 2\mu\boldsymbol{e}$$

$$\text{for i} \neq \text{j},\ T_{ij} = 2\mu e_{ij} \quad \Rightarrow \quad \boxed{G = \mu}$$

- Bulk modulus, K = average normal stress divided by volume change, Δ (dilation):

$$\frac{1}{3}\mathrm{tr}(\boldsymbol{T}) = \frac{1}{3}T_{kk} = \lambda(e_{11}+e_{22}+e_{33}) + \frac{2}{3}\mu(e_{11}+e_{22}+e_{33})$$

$$\Delta := tr(e) = e_{kk} = (e_{11}+e_{22}+e_{33})$$

$$\boxed{K := \frac{1/3\,\mathrm{tr}(\boldsymbol{T})}{\Delta} = \frac{3\lambda+2\mu}{3}}$$

Generalized Hooke's Law

- 36 coefficients in elasticity matrix
- Additional required symmetry ⇒ 21 coefficients
- Orthotropy - properties are different in three orthogonal directions ⇒ 9 coefficients

$$[\mathbf{T}] = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{13} \end{bmatrix},\ [\mathbf{e}] = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix},\ [\mathrm{C}] = \begin{bmatrix} \mathrm{C}_{1111} & \mathrm{C}_{1122} & \mathrm{C}_{1133} & 0 & 0 & 0 \\ & \mathrm{C}_{2222} & \mathrm{C}_{2233} & 0 & 0 & 0 \\ & & \mathrm{C}_{3333} & 0 & 0 & 0 \\ & & & \mathrm{C}_{1212} & 0 & 0 \\ & \text{(symmetric)} & & & \mathrm{C}_{2323} & 0 \\ & & & & & \mathrm{C}_{1313} \end{bmatrix}$$

Generalized Hooke's Law (cont'd)

- The inverse of the elasticity matrix for an *orthotropic material* can be written as:

$$[e]=[C]^{-1}\cdot[T] \Rightarrow [C]^{-1}=\begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{31}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{32}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{13}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{23}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{13}} \end{bmatrix}$$

- Note that there are 12 coefficients identified here, but we said that only 9 were unique. The Maxwell-Betti Reciprocity theorem (see handout) leads to the additional relations between the material coefficients, namely:

$$\frac{\nu_{12}}{E_1}=\frac{\nu_{21}}{E_2}; \quad \frac{\nu_{13}}{E_1}=\frac{\nu_{31}}{E_3}; \quad \frac{\nu_{23}}{E_2}=\frac{\nu_{32}}{E_3};$$

Generalized Hooke's Law (cont'd)

- For a *transversely isotropic material*, assume that the material is reinforced in the 1 direction and the behavior along the 2 and 3 directions is identical.

$$E_2=E_3, \quad \nu_{12}=\nu_{13}, \quad G_{12}=G_{13}$$

$$[e]=[C]^{-1}\cdot[T] \Rightarrow [C]^{-1}=\begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{12}}{E_1} & \frac{-\nu_{12}}{E_1} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{23}}{E_2} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$

- There are 6 coefficients in the above matrix, but only 5 are unique. By simplifying the equations from the previous slide for orthotropic elasticity to account for the equal entries in the matrix above, we can show:

$$\frac{1}{G_{23}}=\frac{2(1+\nu_{23})}{E_2}$$